

On the parity of the number of nodal domains for an eigenfunction of the Laplacian on tori

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Abstract

In this note, we discuss a question posed by T. Hoffmann-Ostenhof (see [3]) concerning the parity of the number of nodal domains for a non-constant eigenfunction of the Laplacian on flat tori. We present two results. We first show that on the torus $(\mathbb{R}/2\pi\mathbb{Z})^2$, a non-constant eigenfunction has an even number of nodal domains. We then consider the torus $(\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\rho\pi\mathbb{Z})$, with $\rho = \frac{1}{\sqrt{3}}$, and construct on it an eigenfunction with three nodal domains.

Keywords. Laplacian, torus, nodal domains.

MSC classification. 35P99, 35J05.

1 Introduction

We consider the non-negative Laplace-Beltrami operator $-\Delta$ on the torus $\mathbb{T}_\rho^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\rho\pi\mathbb{Z})$, seen as a two-dimensional Riemannian manifold, with $\rho \in (0, 1]$. The eigenvalues of $-\Delta$ are given by

$$\lambda_{m,n} = m^2 + \frac{n^2}{\rho^2},$$

with $(m, n) \in \mathbb{N}^2$, and an associated basis of eigenfunctions is given, in the standard coordinates, by

$$u_{m,n}^{cc}(x_1, x_2) = \cos(mx_1) \cos\left(\frac{nx_2}{\rho}\right);$$

$$u_{m,n}^{cs}(x_1, x_2) = \cos(mx_1) \sin\left(\frac{nx_2}{\rho}\right);$$

$$u_{m,n}^{sc}(x_1, x_2) = \sin(mx_1) \cos\left(\frac{nx_2}{\rho}\right);$$

$$u_{m,n}^{ss}(x_1, x_2) = \sin(mx_1) \sin\left(\frac{nx_2}{\rho}\right).$$

To be more precise, the family consisting of all the above functions that are non-zero is an orthogonal basis of $L^2(\mathbb{T}_\rho^2)$. Let us note that the eigenspace associated with the eigenvalue λ is spanned by all the functions in this basis such that the corresponding pair of indices (m, n) satisfies $\lambda = m^2 + \frac{n^2}{\rho^2}$. If ρ^2 is a rational number, a large eigenvalue can have a very high multiplicity, and an associated eigenfunction can possess a very complex nodal structure (see for instance [2]).

We recall that for any eigenfunction u of $-\Delta$, we call *nodal set* the closed set $\mathcal{N}(u) = u^{-1}(\{0\})$ and *nodal domain* a connected component of $\mathbb{T}_\rho^2 \setminus \mathcal{N}(u)$. We will prove the following statements.

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Theorem 1. *If ρ^2 is irrational or $\rho = 1$, any non-constant eigenfunction u of $-\Delta$ has an even number of nodal domains. More precisely, we can divide the nodal domains of u into pairs of isometric domains, u being positive on one domain of each pair and negative on the other.*

Proposition 2. *If $\rho = \frac{1}{\sqrt{3}}$, there exists an eigenfunction of $-\Delta$ with three nodal domains.*

In [3], T. Hoffmann-Ostenhof asked if there exists a torus $(\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\rho\mathbb{Z})$, with $\rho \in (0, 1]$, for which some eigenfunction of the Laplacian has an odd number of nodal domains, at least equal to three. Proposition 2 answers the question positively, while Theorem 1 shows that such an eigenfunction does not exist when ρ^2 is irrational or $\rho = 1$.

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2 Proof of the theorem

Let us outline the method we will use to prove Theorem 1. Let us first note that to any vector $v = (v_1, v_2) \in \mathbb{R}^2$, we can associate a bijection $x \mapsto x + v$ from \mathbb{T}_ρ^2 to itself. It is defined in the following way: if $x = (x_1, x_2)$ in the standard coordinates, $x + v = (x_1 + v_1 \bmod 2\pi, x_2 + v_2 \bmod 2\rho\pi)$. We will prove the following result.

Proposition 3. *If ρ^2 is irrational or $\rho = 1$, and if u is a non-constant eigenfunction of $-\Delta$ on \mathbb{T}_ρ^2 , there exists $v_u \in \mathbb{R}^2$ such that $u(x + v_u) = -u(x)$ for all $x \in \mathbb{T}_\rho^2$.*

Let us show that Proposition 3 implies Theorem 1. An eigenfunction u being given, we define the bijection $\sigma : x \mapsto x + v_u$ from \mathbb{T}_ρ^2 to itself. It is an isometry that preserves $\mathcal{N}(u)$, and exchanges the nodal domains on which u is positive with those on which u is negative. This proves Theorem 1.

Let us now turn to the proof of Proposition 3. Let us first consider the case where ρ^2 is irrational, and let λ be a non-zero eigenvalue of $-\Delta$. Since ρ^2 is irrational there exists a unique pair of integers (m, n) , different from $(0, 0)$, such that $\lambda = m^2 + \frac{n^2}{\rho^2}$. The eigenspace associated with λ is therefore spanned by the functions $u_{m,n}^{cc}$, $u_{m,n}^{cs}$, $u_{m,n}^{sc}$, and $u_{m,n}^{ss}$. Let us assume that $m > 0$ and let us set $v = (\pi/m, 0)$. It is then immediate to check that, for all x in \mathbb{T}_ρ^2 , $u(x + v) = -u(x)$ when u is any of the basis functions $u_{m,n}^{cc}$, $u_{m,n}^{cs}$, $u_{m,n}^{sc}$, and $u_{m,n}^{ss}$. As a consequence we still have $u(x + v) = -u(x)$ when u is any linear combination of the previous basis functions, that is to say any eigenfunction associated with λ . If $m = 0$, we have $n > 0$ and the same holds true with $v = (0, \rho\pi/n)$. This concludes the proof of Proposition 3 in the irrational case.

Let us now consider the case $\rho = 1$. As in the previous case, we will prove a statement that is slightly more precise than Proposition 3: we will exhibit, for any non-zero eigenvalue λ , a vector $v \in \mathbb{R}^2$ such that $u(x + v) = -u(x)$ for every eigenfunction u associated with λ (see Lemma 5). The difference in this case is that the equality $\lambda = m^2 + n^2$ can be satisfied for several pairs of integers (m, n) . To overcome this difficulty, we will need the following simple arithmetical lemma. This result is stated and proved in [4], where it is used to solve a closely related problem: proving that a non-constant eigenfunction of the Laplacian on the square with a Neumann or a periodic boundary condition must take the value 0 on the boundary. We nevertheless give a proof of the lemma here for the sake of completeness.

Lemma 4. *Let (m, n) be a pair of non-negative integers, with $(m, n) \neq (0, 0)$, and let us write $\lambda = m^2 + n^2$. If $\lambda = 2^{2p}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $m = 2^p m_0$ and $n = 2^p n_0$, where exactly one of the integers m_0 and n_0 is odd. If on the other hand $\lambda = 2^{2p+1}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $n = 2^p m_0$ and $m = 2^p n_0$, where both integers m_0 and n_0 are odd.*

Proof. From the decomposition into prime factors, we deduce that we can write any positive integer N as $N = 2^t N_1$, with t a non-negative and N_1 an odd integer. Let us first consider the case where m or n is zero. Without loss of generality, we can assume that $n = 0$. We write $m = 2^r m_1$. We are in the case $\lambda = 2^{2p}(2q + 1)$ with $p = r$ and $2q + 1 = m_1^2$, and we obtain the desired result by setting $m_0 = m_1$

(odd) and $n_0 = 0$ (even). We now assume that both m and n are positive. We write $m = 2^r m_1$ and $n = 2^s n_1$ with m_1 and n_1 odd integers. Without loss of generality, we can assume that $r \leq s$. We find $\lambda = 2^{2r}(m_1^2 + 2^{2(s-r)}n_1^2)$. If $r < s$, then $m_1^2 + 2^{2(s-r)}n_1^2$ is an odd integer, and we have $\lambda = 2^{2p}(2q+1)$, with $p = r$ and $2q+1 = m_1^2 + 2^{2(s-r)}n_1^2$. In that case, we set $m_0 = m_1$ (odd) and $n_0 = 2^{s-r}n_1$ (even). If $r = s$, we find $\lambda = 2^{2r}(m_1^2 + n_1^2)$. We have furthermore $m_1 = 2m_2 + 1$ and $n_1 = 2n_2 + 1$, and therefore $m_1^2 + n_1^2 = 4(m_2^2 + n_2^2 + m_2 + n_2) + 2$. We have $\lambda = 2^{2p+1}(2q+1)$ with $p = r$ and $q = m_2^2 + n_2^2 + m_2 + n_2$, and we set $m_0 = m_1$ and $n_0 = n_1$. \square

Lemma 5. *Let λ be a non-zero eigenvalue of $-\Delta$ on \mathbb{T}_1^2 .*

- i. *If $\lambda = 2^{2p}(2q+1)$, we set $v = (\pi/2^p, \pi/2^p)$, and we have $u(x+v) = -u(x)$ for every eigenfunction u associated with λ .*
- ii. *If $\lambda = 2^{2p+1}(2q+1)$, we set $v = (\pi/2^p, 0)$, and we have $u(x+v) = -u(x)$ for every eigenfunction u associated with λ .*

Proof. Let us first consider the case where $\lambda = 2^{2p}(2q+1)$. Let us choose a pair of indices (m, n) such that $\lambda = m^2 + n^2$, and let us consider one of the associated basis functions given in the introduction, say $u_{m,n}^{cc}(x, y) = \cos(mx) \cos(ny)$ to fix the ideas. According to Lemma 4, we have $m = 2^p m_0$ and $n = 2^p n_0$, where exactly one of the integers m_0 and n_0 is odd. We can assume, without loss of generality, that m_0 is odd and n_0 even. Then

$$\begin{aligned} \cos\left(m\left(x_1 + \frac{\pi}{2^p}\right)\right) &= \cos(mx_1 + m_0\pi) = -\cos(mx_1), \\ \cos\left(n\left(x_2 + \frac{\pi}{2^p}\right)\right) &= \cos(nx_2 + n_0\pi) = \cos(nx_2), \end{aligned}$$

and therefore

$$u_{m,n}^{cc}(x+v) = -u_{m,n}^{cc}(x).$$

We show in the same way that $u_{m,n}^{cs}(x+v) = -u_{m,n}^{cs}(x)$, $u_{m,n}^{sc}(x+v) = -u_{m,n}^{sc}(x)$, and $u_{m,n}^{ss}(x+v) = -u_{m,n}^{ss}(x)$. Since v depends only on λ , we have $u(x+v) = -u(x)$ as soon as u is a basis function associated with λ , and therefore, by linear combination, as soon as u is an eigenfunction associated with λ .

The case $\lambda = 2^{2p+1}(2q+1)$ can be treated in the same way, taking $v = (\pi/2^p, 0)$ ($v = (0, \pi/2^p)$ would also be suitable). \square

Remark 6. *It can also be shown that Lemma 4 still holds if we replace the equation $\lambda = m^2 + n^2$ by $\lambda = \alpha m^2 + \beta n^2$, where α and β are odd integers such that $\alpha + \beta = 2 \pmod{4}$. This implies that the conclusion of Theorem 1 still holds if $\rho = \sqrt{\frac{\alpha}{\beta}}$, with α and β as above.*

3 Proof of the proposition

In this section, we assume that $\rho = \frac{1}{\sqrt{3}}$. Let us outline the idea we will use to construct the eigenfunction whose existence is asserted in Proposition 2. It will belong to the eigenspace associated with the eigenvalue 4. We start from the eigenfunction

$$u_{1,1}^{cc}(x_1, x_2) = \cos(x_1) \cos(\sqrt{3}x_2),$$

which has four rectangular nodal domains, shown in Figure 1(a). We perturb this eigenfunction by adding a small multiple of the eigenfunction

$$u_{2,0}^{cc}(x_1, x_2) = \cos(2x_1).$$

For $\varepsilon > 0$, we get the eigenfunction

$$v_\varepsilon(x_1, x_2) = u_{1,1}^{cc}(x_1, x_2) + \varepsilon u_{2,0}^{cc}(x_1, x_2).$$

Since $u_{2,0}^{cc}$ is negative in the neighborhood of the critical points in $\mathcal{N}(u_{1,1}^{cc})$, adding $\varepsilon u_{2,0}^{cc}$ has the effect of opening small "channels" that connect the nodal domains where $u_{1,1}^{cc}$ is negative. As a result, if $\varepsilon > 0$

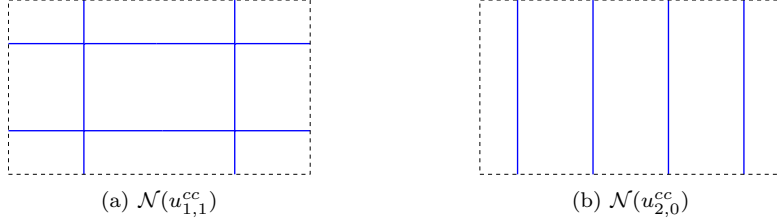


Figure 1: Nodal sets of basis functions

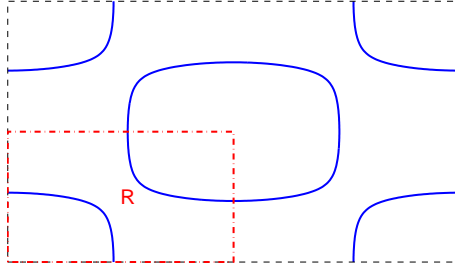


Figure 2: Nodal set of $v_\epsilon = u_{1,1}^{cc} + \epsilon u_{2,0}^{cc}$ with $\epsilon = 0.1$

is small enough, v_ϵ has three nodal domains, one where it is negative and two where it is positive (see Figure 2). Let us note that these ideas have already been used, to construct examples of eigenfunctions whose nodal set satisfies some prescribed properties, in [6, 5, 1]. In particular, the desingularization of critical points in the nodal set, that we have briefly described, is studied in details in [1, 6.7]

To prove rigorously these assertions, let us consider the open domain R in \mathbb{T}_ρ^2 defined, in the standard coordinates (x_1, x_2) , as

$$R =]0, \pi[\times]0, \frac{\pi}{\sqrt{3}}[.$$

We now define (following [1]) the smooth change of coordinates

$$\begin{cases} \xi_1 &= -\cos(x_1), \\ \xi_2 &= -\cos(\sqrt{3}x_2), \end{cases}$$

which sends R to $] -1, 1[\times] -1, 1[$. In these new coordinates the nodal set of the function v_ϵ satisfies the equation

$$\xi_1 \xi_2 + \epsilon(2\xi_1^2 - 1) = 0.$$

A simple computation shows that this is the equation of an hyperbola. Furthermore, when $0 < \epsilon < 1$, this hyperbola has one branch in the lower left quadrant $] -1, 0[\times] -1, 0[$ and one in the upper right quadrant $]0, 1[\times]0, 1[$.

On the other hand, we have the following symmetries of v_ϵ :

$$v_\epsilon(2\pi - x_1, x_2) = v_\epsilon(x_1, x_2)$$

and

$$v_\epsilon\left(x_1, \frac{2\pi}{\sqrt{3}} - x_2\right) = v_\epsilon(x_1, x_2).$$

This allows us to recover the nodal set on the whole of \mathbb{T}_ρ^2 . It consists in two simple closed curves, each containing one branch of the previously considered hyperbola. Each of these closed curves enclose a nodal domain that is homeomorphic to a disk. Let us now consider the complement of the closure of those two region. It is the third nodal domain.

Remark 7. *The construction used to prove Proposition 2 can obviously be generalized. We can for instance consider the eigenfunction*

$$v_\varepsilon(x_1, x_2) = u_{m,n}^{c,c}(x_1, x_2) + \varepsilon u_{km,0}^{c,c}(x_1, x_2),$$

assuming that

$$m^2 + \frac{n^2}{\rho^2} = k^2 m^2.$$

We have in that case $\rho = \frac{n}{m\sqrt{k^2-1}}$. Following the same line of reasoning as in this section, we see that for $\varepsilon > 0$ small enough, v_ε has $2mn + 1$ nodal domains.

In view of Remarks 6 and 7, it would be desirable to obtain a characterization of the rational numbers q , such that there exists an eigenfunction of $-\Delta$ on the torus $\mathbb{T}_{\sqrt{q}}^2$ with an odd number of nodal domains. Unfortunately, we have not been able to reach this goal so far.

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