Eigenvalues variations for Aharonov-Bohm operators

Corentin Léna

UNITO

22 May 2015

Plan

Plan

Magnetic potential and magnetic field

- Magnetic potential : $X_i \in \mathbb{R}^2$ and $\alpha_i \in \mathbb{R}$,

$$\mathbf{A}_{\alpha_i}^{X_i} = \frac{\alpha_i}{r}\mathbf{e}_\theta$$

with polar coordinates (r, θ) referred to X_i.

- Magnetic field :

$$B := \text{Curl } \mathbf{A}_{\alpha_i}^{X_i} = \partial_1 A_2 - \partial_2 A_1$$

- For a closed path γ going once around X_i in the clockwise direction,

$$\frac{1}{2\pi}\oint_\gamma \mathbf{A}_{\alpha_i}^{X_i} \cdot d\mathbf{x} = \alpha_i.$$

In $\mathbb{R}^2 \setminus \{X_i\}$, $B = 0$.

Aharonov-Bohm operators

- $\mathbf{X} = (X_1, \dots, X_N)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$;
- $\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}} = \sum_{i=1}^{N} \mathbf{A}_{\alpha_i}^{X_i}$;
- Ω open, bounded and piecewise C^1 and $\Omega_{\mathbf{X}} := \Omega \setminus \{X_1, \dots, X_N\}$;
- $H_{\mathbf{X}} := \left(i\nabla + \mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}\right)^2$;
- Eigenvalue problem :

$$\begin{cases} H_{\mathbf{X}} u &= \lambda u \text{ in } \Omega_{\mathbf{X}}; \\ u &= 0 \text{ on } \partial\Omega; \end{cases}$$

- Sequence of eigenvalues : $(\lambda_k(\mathbf{X}, \boldsymbol{\alpha}))$.

Motivation : minimal partitions

The domain Ω and an integer $k \geq 1$ are given. We consider $\mathcal{D} = (D_1, \dots, D_k)$ with $D_i \cap D_j = \emptyset$. We try to solve the following optimization problem :

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}} \left\{ \max_{1 \leq i \leq k} \lambda_1(D_i) \right\}.$$

Minimal partitions for this problem exist and are very regular. In particular, if $\mathcal{D}$ is minimal

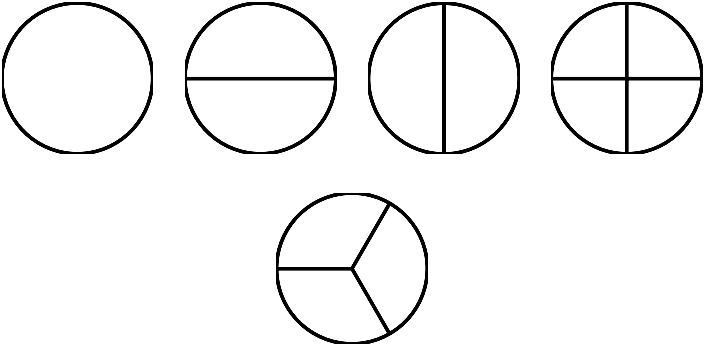
$$\lambda_1(D_1) = \cdots = \lambda_1(D_k).$$

(Bucur–Buttazzo–Henrot 1998, Conti–Terracini–Verzini 2005, Caffarelli–Lin 2007, Helffer–Hoffmann-Ostenhof–Terracini 2009)

If a minimal partition can be colored with only two colors, it is a nodal partition (for the Dirichlet Laplacian). This occurs only for an eigenfunction associated with $\lambda_k(\Omega)$ (Courant-sharp situation). Conversly, if an eigenfunction associated with $\lambda_k(\Omega)$ has k nodal domains, they realize a minimal partition.

(Helffer–Hoffmann-Ostenhof–Terracini, 2009)

Example : the disk



Theorem (Helffer– Hoffmann-Ostenhof, 2013)

Let us assume that $\mathcal{D} = \{D_1, \dots, D_k\}$ is a minimal k-partition of Ω. There exist a finite number of points $X_1, \dots, X_N$ in $\mathbb{R}^2$ such that $\mathcal{D}$ is the nodal partition associated with an eigenfunction u of the operator $H_{\mathbf{X}}$, with $\mathbf{X} = (X_1 \dots, X_N)$ and $\alpha = (1/2, \dots, 1/2)$.
Furthermore, the eigenfunction u is associated with the eigenvalue $\lambda_k(\mathbf{X}, \alpha)$.

To build the magnetic potential, we have to add poles :

- at each singular point of the boundary of $\mathcal{D}$ where an odd number of curves meet ;
- in each hole with an odd number of curves touching its boundary.

Applications :

- $\mathfrak{L}_k(\Omega) = \inf_{N \geq 0} \inf_{\mathbf{X}=(X_1,\dots,X_N)} L_k(\Omega_{\mathbf{X}})$
- numerical search for minimal partitions (Bonnaillie-Noël–Helffer–Hoffmann-Ostenhof–Vial) ;
- the number of odd multiple points tends to $+\infty$ as $k \to +\infty$ (Helffer, 2015).

Plan

Quadratic form and Friedrichs extension

- For $u \in C_c^\infty(\Omega_{\mathbf{X}})$, $q_{\mathbf{X}}(u) := \int \left|\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u\right|^2 dx$.
- $Q_{\mathbf{X}}$ is the completion of $C_c^\infty(\Omega_{\mathbf{X}})$ under the norm associated with $q_{\mathbf{X}}$.
- According to Friedrichs extension theorem, there is a unique positive self-adjoint extension of $\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right)^2$ (differential operator acting on $C_c^\infty(\Omega_{\mathbf{X}})$) with domain contained in $Q_{\mathbf{X}}$.
- This extension is called the Aharonov-Bohm operator and is denoted by $H_{\mathbf{X}}$.

Gauge invariance

- A gauge transformation on $\Omega_{\mathbf{X}}$ acts on pairs vector field-function as $(\mathbf{A}, u) \mapsto (\mathbf{A}^*, u^*)$, with

$$\begin{cases} \mathbf{A}^* &= \mathbf{A} + \nabla\varphi\,, \\ u^* &= e^{i\varphi} u\,, \end{cases}$$

  where φ is a real-valued function on $\Omega_{\mathbf{X}}$ (possibly multivalued).
- A gauge transformation does not change the magnetic field $\mathbf{B} = \mathrm{Curl}\,\mathbf{A}$, nor the probability distribution $|u|^2$.
- If $\mathbf{A}$ and $\mathbf{A}^*$ are two gauge equivalent magnetic potentials in $C^\infty(\Omega_{\mathbf{X}}, \mathbb{R}^2)$, the operators $H_{\mathbf{A}}$ and $H_{\mathbf{A}^*}$ are unitarily equivalent.
- The potential $\mathbf{A}$ and $\mathbf{A}'$ are gauge equivalent, if and only if,

$$\frac{1}{2\pi} \oint_\gamma \left(\mathbf{A}'(x) - \mathbf{A}(x)\right) d\mathbf{x}$$

  is an integer for any loop γ contained in $\Omega_{\mathbf{X}}$. (Helffer–Hoffmann-Ostenhof, M.&T.–Owen, 1999)

Hardy inequality

Proposition (Laptev–Weidl, 1998, Alziary–Fleckinger-Pellé–Takáč, 2003)

If $\alpha_i \notin \mathbb{Z}$, $\rho > 0$,and $u \in C^\infty(\Omega \setminus \{X_i\})$,

$$\int_{|x-X_i|<\rho} \frac{|u|^2}{|x-X_i|^2}\,dx \leq C \int_{|x-X_i|<\rho} \left|\left(i\nabla + \mathbf{A}_{\alpha_i}^{X_i}\right)u\right|^2 dx,$$

where

$$C := \frac{1}{\inf_{n\in\mathbb{Z}} |n-\alpha_i|^2}.$$

Corollary

If $X_i \neq X_j$ and $\alpha_i \notin \mathbb{Z}$, then $Q_{\mathbf{X}} \subset H_0^1(\Omega)$.

Proof

Use polar coordinates centered at X_i

$$\left|\left(i\nabla + \mathbf{A}_{\alpha_i}^{X_i}\right) u\right|^2 = |\partial_r u|^2 + \frac{1}{r^2}\left|(i\partial_\theta + \alpha)\, u\right|^2.$$

$$\int_{B(X_i,\rho)} \left|\left(i\nabla + \mathbf{A}_{\alpha_i}^{X_i}\right) u\right|^2 dxdy \geq \int_0^\rho rdr \int_0^{2\pi} \frac{d\theta}{r^2} \left|(i\partial_\theta + \alpha)\, u\right|^2.$$

We write

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\theta} \text{ then } \partial_\theta u(r,\theta) = \sum_{n\in\mathbb{Z}} -inc_n(r)e^{in\theta}.$$

According to Parseval's Formula ,

$$\int_{B(X_i,\rho)} \frac{|u|^2}{r^2}\, dxdy = 2\pi \int_0^\rho \frac{1}{r^2}\left(\sum_{n\in\mathbb{Z}} |c_n(r)|^2\right) r\, dr$$

and

$$\int_0^\rho rdr \int_0^{2\pi} \frac{d\theta}{r^2} \left|(i\partial_\theta + \alpha)\, u\right|^2 = 2\pi \int_0^\rho \frac{1}{r^2}\left(\sum_{n\in\mathbb{Z}} |n-\alpha|^2 |c_n(r)|^2\right) r\, dr.$$

Characterization of the form domain

If $u \in L^2(\Omega)$, $\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u \in \mathcal{D}'(\Omega_{\mathbf{X}}, \mathbb{C}^2)$. We define

$$\mathcal{H}_{\mathbf{X}}(\Omega) := \left\{u \in L^2(\Omega) \,:\, \left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u \in L^2(\Omega)\right\}.$$

Proposition

i. $\mathcal{H}_{\mathbf{X}}(\Omega) \subset L^2(\Omega)$ *compactly*;

ii. *there is a trace operator* $\gamma_0 : \mathcal{H}_{\mathbf{X}}(\Omega) \to L^2(\partial\Omega)$;

iii. $u \in Q_{\mathbf{X}}$ *if, and only if,* $u \in \mathcal{H}_{\mathbf{X}}(\Omega)$ *and* $\gamma_0 u = 0$.

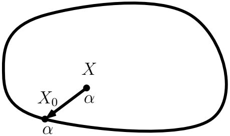
Continuity and consequences

Theorem

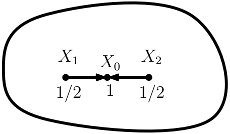
The function $\mathbf{X} \mapsto \lambda_k(\mathbf{X}, \boldsymbol{\alpha})$ *is continuous.*

Applications :

- $\lambda_k(X,\alpha) \to \lambda_k(X_0,\alpha) = \lambda_k(\Omega)$;



- $\lambda_k(X_1, X_2, 1/2, 1/2) \to \lambda_k(X_0, 1) = \lambda_k(\Omega)$.



Plan

Non-concentration inequality

Proposition

There exists C *such that for all* $x_0 \in \Omega$ *and* $r > 0$,

$$\|u\|^2_{L^2(B(x_0,r))} \leq Cr\|\nabla u\|^2_{L^2(\Omega)} \text{ for all } u \in H^1_0(\Omega)$$

and

$$\|u\|^2_{L^2(B(x_0,r))} \leq Cr\left\|\left(i\nabla + \mathbf{A}^{\mathbf{X}}_\alpha\right) u\right\|^2_{L^2(\Omega)} \text{ for all } u \in Q_{\mathbf{X}}$$

Proof : Sobolev injection, diamagnetic inequality.

Main Lemma

Let $\mathbf{X}^n \to \mathbf{X} \in \mathbb{R}^{2N}$ ($X_i^n \to X_i$ for all $1 \leq i \leq N$).

Lemma

If

- $H_{\mathbf{X}^n} u^n = \lambda^n u^n$ *with* $u^n \in Q_{\mathbf{X}^n}$,
- $\|u^n\|_{L^2(\Omega)} = 1$,
- $\lambda^n \to \lambda$,

then there exist a subsequence (λ^{n_p}, u^{n_p}) *and* $u \in Q_{\mathbf{X}}$ *such that*

- $u^{n_p} \to u$ *strongly in* $L^2(\Omega)$ *and almost everywhere in* Ω,
- $H_{\mathbf{X}} u = \lambda u$.

We define

$$S_m := \bigcup_{i=1}^{N} \overline{B(X_i, \frac{1}{m})};$$

and

$$\Omega_m := \Omega \setminus S_m; .$$

Then $(u^n_{|\Omega_m}) \subset \mathcal{H}_{\mathbf{X}}(\Omega_m)$ for n large enough, bounded.
By compact injection, there exists a subsequence converging

- weakly in $\mathcal{H}_{\mathbf{X}}(\Omega_m)$;
- strongly in $L^2(\Omega_m)$;
- almost everywhere in Ω_m.

By diagonal extraction, we find a subsequence (u^{n_p}) converging

- almost everywhere on Ω ;
- weakly in $\mathcal{H}_{\mathbf{X}}(\Omega_m)$ and strongly in $L^2(\Omega_m)$ for all m.

We define $u(x) := lim_{p \to +\infty} u^{n_p}(x)$ almost everywhere.

Proof of the lemma II

$$\int_{\Omega_m} |u^{n_p}|^2 \leq \int_{\Omega} |u^{n_p}|^2 = 1 \text{ therefore } u \in L^2(\Omega).$$

$$\int_{\Omega_m} \left|\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u^{n_p}\right|^2 \leq 2\int_{\Omega_m} \left|\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}^{n_p}}\right) u\right|^2 + \int_{\Omega_m} \left|\mathbf{A}_\alpha^{\mathbf{X}^{n_p}} - \mathbf{A}_\alpha^{\mathbf{X}}\right|^2 |u^{n_p}|^2$$

and therefore

$$\int_{\Omega_m} \left|\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u\right|^2 \leq \sup_{q \geq 1} \lambda^q \text{ for all } m.$$

Therefore $\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u$ (which is in $\mathcal{D}'\left(\Omega_{\mathbf{X}}, \mathbb{C}^2\right)$) is in $L^2(\Omega)$. We can show that $\gamma_0 u = 0$, therefore $u \in Q_{\mathbf{X}}$. We have easily $H_{\mathbf{X}} u = \lambda u$. It remains to show that $u \neq 0$. By the non-concentration inequality,

$$\int_{S_m} |u^{n_p}|^2 \leq \frac{CN}{m} \int_{\Omega} \left|\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}\right) u^{n_p}\right|^2 \leq \frac{CN}{m} \sup_{q \geq 1} \lambda^q.$$

From this we deduce that $u^{n_p} \to u$ strongly in $L^2(\Omega)$, in particular $\|u\|_{L^2(\Omega)} = 1$.

Proof of the result

Lemma

$$\limsup_{n\to+\infty} \lambda_k(\mathbf{X}^n, \alpha) \le \lambda_k(\mathbf{X}, \alpha).$$

Proof : min-max formula

$$\lambda_k(\mathbf{X}, \alpha) = \inf_{\varphi_1,\dots,\varphi_k \in C_c^\infty(\Omega_{\mathbf{X}})} \ \max_{u \in vect(\varphi_1,\dots,\varphi_k)} \frac{\|(-i\nabla - \mathbf{A}_\alpha^{\mathbf{X}})u\|^2}{\|u\|^2}.$$

Let us consider the first eigenvalue. The first lemma implies that

$$\liminf_{n\to+\infty} \lambda_1(\mathbf{X}^n, \alpha) \ge \lambda_1(\mathbf{X}, \alpha).$$

The second lemma then give

$$\lim_{n\to+\infty} \lambda_1(\mathbf{X}^n, \alpha) = \lambda_1(\mathbf{X}, \alpha).$$

We prove the general theorem by induction.

Plan

Statement of the result

We assume that $\mathbf{X}$ is such that $X_i \neq X_j$ and $X_i \notin \partial\Omega$. We write

$$\mathbf{t} = (t_1, t_2, \ldots, t_{2N-1}, t_{2N}) \in \mathbb{R}^{2N}$$

and

$$\mathbf{X}(\mathbf{t}) := (X_1 + (t_1, t_2), \ldots, X_N + (t_{2N-1}, t_{2N})).$$

Theorem

If $\lambda_k(\mathbf{X}, \alpha)$ is simple , the function $\mathbf{t} \mapsto \lambda_k(\mathbf{X}(\mathbf{t}), \alpha)$ is analytic in a neighborhood of 0.

(Bonnaillie-Noël–Noris–Nys–Terracini, 2013)

Analytic family of forms

We define $V_{\mathbf{t}}$ a vector field with value :

- (t_{2i-1}, t_{2i}) at X_i ;
- zero outside of a small neighborhood of the X_i's.

We define :

- $\Phi_{\mathbf{t}} : x \mapsto x + V_{\mathbf{t}}(x)$;
- $$\begin{array}{rccc} U_{\mathbf{t}} : & C_c^\infty(\Omega_{\mathbf{X}}) & \to & C_c^\infty(\Omega_{\mathbf{X}(\mathbf{t})}) \\ & L^2(\Omega) & \to & L^2(\Omega) \\ & u & \mapsto & \sqrt{J\left(\Phi_{\mathbf{t}}^{-1}\right)}\, u \circ \Phi_{\mathbf{t}}^{-1}; \end{array}$$
- $r_{\mathbf{t}}(u) := q_{\mathbf{X}(\mathbf{t})}\left(U_{\mathbf{t}} u\right)$ for all $u \in C_c^\infty(\Omega_{\mathbf{X}})$.

Direct estimates shows that there exist $0 < a < 1$ and $b \geq 0$ such that

$$|r_{\mathbf{t}}(u) - q_{\mathbf{X}}(u)| \leq a q_{\mathbf{X}}(u) + b\|u\|^2_{L^2(\Omega)}.$$

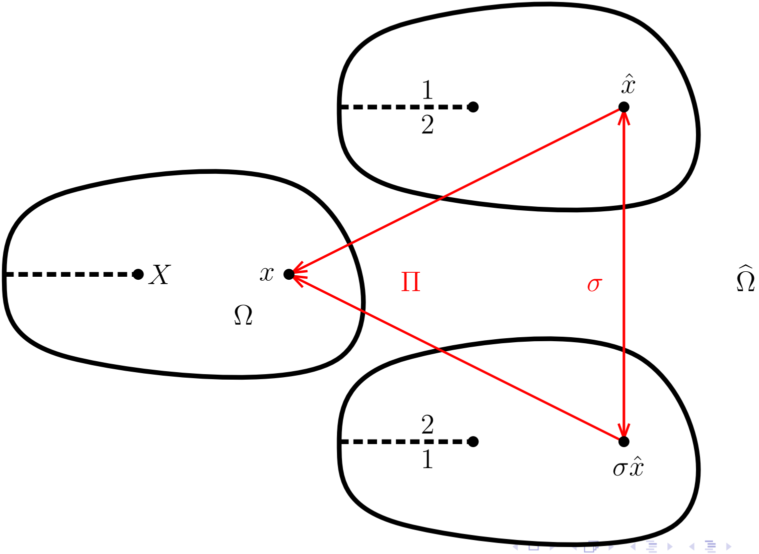
According to Kato's theory on analytic families of quadratic forms, we have the conclusion of the theorem.

Plan

Conjugation operator (Helffer–Hoffmann-Ostenhof, M.&T.–Owen, 1999)

- We assume that $\mathbf{X}$ is such that $X_i \neq X_j$ and $X_i \notin \partial\Omega$, and furthermore that $\alpha_i \in \frac{1}{2} + \mathbb{Z}$ for all $1 \leq i \leq N$.
- In that case, we can identify a class of functions for which the notion of nodal set is meaningful.
- We have $2\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq N$, therefore there exists φ such that $\nabla\varphi = 2\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}$.
- Unitary antilinear operator : $K_{\mathbf{X}} : u \mapsto e^{i\varphi}\overline{u}$.
- $H_{\mathbf{X}} \circ K_{\mathbf{X}} = K_{\mathbf{X}} \circ H_{\mathbf{X}}$.
- Definition of a $K_{\mathbf{X}}$-real function : $K_{\mathbf{X}} u = u$.

Geometric interpretation : covering (Helffer–Hoffmann-Ostenhof, M.&T.–Owen, 1999)



Geometric interpretation : antisymmetric functions

(Helffer–Hoffmann-Ostenhof, M.&T.–Owen, 1999)

- $$\begin{array}{cccc} \Sigma : & L^2\left(\widehat{\Omega}\right) & \to & L^2\left(\widehat{\Omega}\right) \\ & u & \mapsto & u\circ\sigma \end{array}$$
- $\mathcal{S} := ker(\Sigma - Id)$ (symmetric functions) and $\mathcal{A} := ker(\Sigma + Id)$ (antisymmetric functions).
- $L^2\left(\widehat{\Omega}\right) = \mathcal{S} \oplus \mathcal{A}$
- $-\widehat{\Delta}$ Laplace-Beltrami operator on $\widehat{\Omega}$, $-\widehat{\Delta}\circ\Sigma = \Sigma\circ\left(-\widehat{\Delta}\right)$.
- The eigenvalues of $-\widehat{\Delta}_{|\mathcal{S}}$ are the eigenvalues of the Dirichlet Laplacian, the eigenvalues of $-\widehat{\Delta}_{|\mathcal{A}}$ are the eigenvalues of H_X with flux $1/2$.
- More precisely, the mapping $u \mapsto \hat{u}e^{-i\hat{\varphi}/2}$ give a correspondence between $K_{\mathbf{X}}$-real eigenfunctions of H_X and real antisymmetric eigenfunctions of $-\widehat{\Delta}$.

Nodal set

Theorem (Alziary– Fleckinger-Pellé–Takáč, 2003)

If u is a $K_{\mathbf{X}}$-real eigenfunction of $H_{\mathbf{X}}$ and X_i a pole, there exist $m \in \mathbb{N}$, f and g C^1-functions such that

- $f(X_i) \neq 0$,
- $u(x) = |x - X_i|^{m+\frac{1}{2}} f(x)$,
- $\left(i\nabla + \mathbf{A}_\alpha^{\mathbf{X}}(x)\right) u(x) = |x - X_i|^{m-\frac{1}{2}} g(x)$,
- $2m + 1$ is the number of nodal lines meeting at X_i.

Critical points

Theorem

Assume that $\lambda_k(\mathbf{X}, \alpha)$ is simple and has a $K_{\mathbf{X}}$-real eigenfunction with at least 3 nodal lines meeting at X_i. Let $\mathbf{v} \in \mathbb{R}^2$,

$$\mathbf{X}(t) := (X_1, \ldots, X_i + t\mathbf{v}, \ldots, X_N) \text{ and } \lambda_k(t) := \lambda_k(\mathbf{X}(t), \alpha).$$

Then

$$\lambda_k'(0) = 0.$$

(Noris–Terracini, 2009, Bonnaillie-Noël–Noris–Nys–Terracini, 2013)

To prove this, we construct a family of diffeomorphisms $\Phi_{h,t}$ that depends on the additional parameter $h > 0$. Using the Feynman-Hellmann formula, we compute $\lambda_k'(0)$ (which does not depend on h) as an integral $I(h)$ depending on h (integral of a function supported on a disk of size h centered at X_i). We then use the local estimates on u, with $m \geq 1$, to show that $\lim_{h \to 0} I(h) = 0$.

Theorem

Let us assume that Ω *is a connected open set,* k *a positive integer, and* $\mathcal{D}$ *a minimal k-partition of* Ω *. We denote by* $\mathbf{X} = (X_1, \dots, X_N)$ *and* $\alpha = (1/2 \ \dots, 1/2)$ *poles and fluxes as defined in the magnetic characterization. Let us additionally assume that the eigenvalue* $\lambda_k(\mathbf{X}, \alpha)$ *is simple. The point* $\mathbf{X}$ *is then critical for the function* $\mathbf{Y} \mapsto \lambda_k(\mathbf{Y}, \alpha)$ *, which is defined and analytic in a neighborhood of* $\mathbf{X}$ *.*

Proof of the theorem

We recall that $\mathbf{Y} \mapsto \lambda_k(\mathbf{Y}, \boldsymbol{\alpha})$ is analytic in a neighborhood of $\mathbf{X}$, and that, according to the magnetic characterization, there exists a $K_{\mathbf{X}}$-real eigenfunction u whose nodal partition is $\mathcal{D}$.
We now show that the gradient of $\mathbf{Y} \mapsto \lambda_k(\mathbf{Y}, \boldsymbol{\alpha})$ with respect to each variable X_i is zero at $\mathbf{X}$.

- If $X_i \in \mathbb{R}^2 \setminus \overline{\Omega}$, we have $\lambda_k(\mathbf{Y}, \boldsymbol{\alpha}) = \lambda_k(\mathbf{X}, \boldsymbol{\alpha})$ for $\mathbf{Y} = (Y_1, \ldots, Y_N)$ such that $Y_j = X_j$ for $j \neq i$ and Y_i is in the same connected component of $\mathbb{R}^2 \setminus \overline{\Omega}$ as X_i (we use a gauge transformation).
- If $X_i \in \Omega$, at least three nodal lines of u meet at X_i. Therefore, according to our results, X_i is a critical point for the function

$$Y \mapsto \lambda_k((X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_N), \boldsymbol{\alpha}).$$

Example : sector with a pole on the axis (Bonnaillie-Noël)

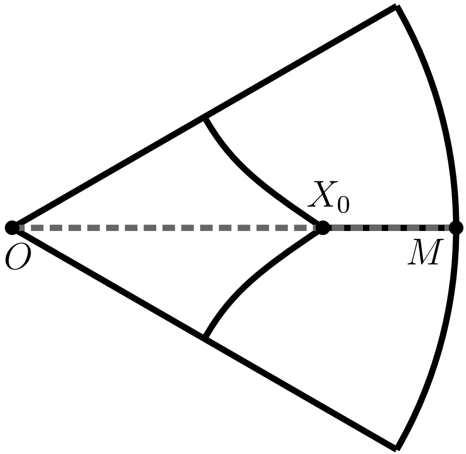


Figure : Aharonov-Bohm problem with symmetry.

Generalization : symmetric domains

(Bonnaillie-Noël–Helffer–Hoffmann-Ostenhof, 2009)

We assume that Ω is simply connected and that the line $\{x_2 = 0\}$ is an axis of symmetry .
We consider an Aharonov-Bohm operator with one pole $X = (x, 0) \in \{x_2 = 0\}$ and $\alpha = 1/2$.
We note $\Omega^+ = \Omega \cap \{x_2 > 0\}$, $\Gamma^+ = \partial\Omega \cap \{x_2 > 0\}$, and $\Omega \cap \{x_2 = 0\} = (O, M)$.
We now consider two eigenvalue problems with mixed boundary conditions.

$$\begin{cases} -\Delta u = \lambda_k^{DN}(x) u \text{ in } \Omega^+ , \\ u = 0 \text{ on } [O, X] \cup \Gamma^+ , \\ \partial_{\mathbf{n}} u = 0 \text{ on } (X, M) ; \end{cases} \qquad \begin{cases} -\Delta u = \lambda_k^{ND}(x) u \text{ in } \Omega^+ , \\ \partial_{\mathbf{n}} u = 0 \text{ on } (O, X) , \\ u = 0 \text{ on } [X, M] \cup \Gamma^+ . \end{cases}$$

The spectrum of $-\Delta^X_{1/2}$ is the reunion (counted with multiplicities) of the sequences $(\lambda_k^{DN}(x))_{k \geq 1}$ and $(\lambda_k^{ND}(x))_{k \geq 1}$. Real eigenfunctions of $-\Delta$ correspond to $K_{\mathbf{X}}$-real eignfunctions of $-\Delta^X_{1/2}$.

Search for a 3-partition (Bonnaillie-Noël)

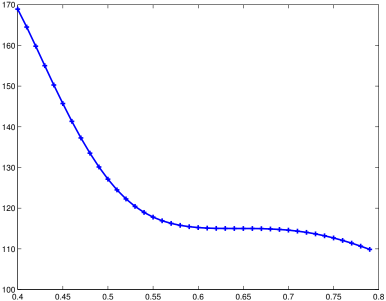


Figure : Eigenvalue $\lambda_3(X, 1/2)$ as a function of x.

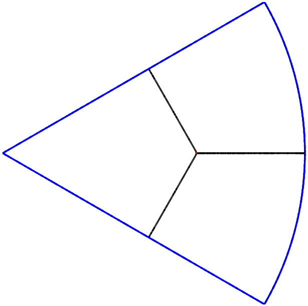


Figure : Nodal set of a third eigenfunction of an Aharonov-Bohm operator.

There is a point of inflexion for $x \simeq 0.64$, that corresponds to three nodal lines meeting at $X = (x, 0)$.