Nodal patterns for the Laplacian on flat tori

Workshop on Variational Perspectives

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Plan

1. Review of general results
2. The square two-dimensional torus
3. The cubic three-dimensional torus
4. Parity of the number of nodal domains on rectangular tori

Plan

Introduction

- M compact and connected manifold (with or without boundary) of dimension n ; this include the case of a bounded open set $\Omega \subset \mathbb{R}^n$.
- $\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \dots$ eigenvalues of the Laplacian $-\Delta_M$, counted with multiplicities : for all $k \geq 1$, there a non-zero eigenspace $E_{\lambda_k(M)}$, such that for all $u \in E_{\lambda_k(M)}$,

$$-\Delta_M u = \lambda_k(M) u.$$

If $\partial M \neq \emptyset$, we impose a boundary condition :

$$u = 0 \text{ on } \partial M \text{ (Dirichlet) or } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M \text{ (Neumann).}$$

For some domains we can make explicit computation of the spectrum and of a basis of eigenfunctions : rectangles, flat tori, spheres and balls, some triangles, circular sectors,. . .

Nodal set and nodal domains

If u eigenfunction :

- nodal set : $\mathcal{N}(u) = \overline{\{x \in M\,;\, u(x) = 0\}}$;
- nodal domain : connected component of $M \setminus \mathcal{N}(u)$;
- nodal partition associated with u : family of all the nodal domains of u.

  What information can we get about these objects ?

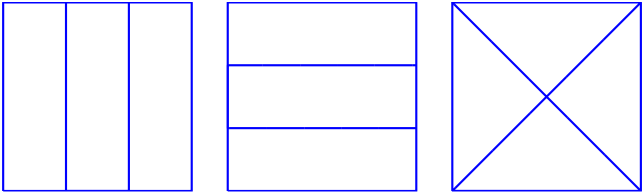
We focus on the number of nodal domains $\nu(u)$, called the nodal count.

For λ eigenvalue, we define

$$\kappa(\lambda) := \min\{k \,:\, \lambda_k(M) = \lambda\}.$$

Example : increase of nodal count by linear combination

- $\Omega = (0,1)^2 \subset R^2$ and $\lambda = 10\pi^2 = \pi^2(3^3 + 1^2)$.
- $u(x,y) = \sin(3\pi x)\sin(\pi y)$ and $v(x,y) = \sin(\pi y)\sin(3\pi x)$.
- If $w := u + v$, $w(x,y) = 4\sin(\pi x)\sin(\pi y)\sin(\pi(x+y))\sin(\pi(y-x))$.
- $\nu(u) = \nu(v) = 3$ and $\nu(u+v) = 4$.



(a) $\mathcal{N}(u)$ (b) $\mathcal{N}(v)$ (c) $\mathcal{N}(w)$

Courant nodal domain theorem

Theorem (Courant, 1923)

If u is an eigenfunction of $-\Delta_M$ associated with the eigenvalue λ, $\nu(u) \leq \kappa(\lambda)$.

Proof : By contradiction, let $\kappa := \kappa(\lambda)$ and let u be an eigenfunction associated with λ, with nodal domains

$$D_1, \dots, D_\kappa, D_{\kappa+1}, \dots$$

There is a non-zero linear combination

$$\varphi = \alpha_1 \varphi_{D_1} + \cdots + \alpha_\kappa \varphi_{D_\kappa}$$

orthogonal to each u_i for $1 \leq i \leq \kappa - 1$. Therefore, according to the max-min principle

$$\lambda_\kappa \leq \frac{\int_M |\nabla \varphi|^2 \, dx}{\int_M |\varphi|^2 \, dx} \leq \lambda = \lambda_\kappa,$$

and φ is an eigenfunction associated to λ. But φ is identically zero on $D_{\kappa+1}$, in contradiction with unique continuation.

Courant-sharp eigenvalue and minimal partitions

Definition

We say that an eigenfunction u associated with the eigenvalue λ is Courant-sharp if $\nu(u) = \kappa(\lambda)$. We say an eigenvalue λ is Courant-sharp if E_λ contains a Courant-sharp eigenfunction.

Definition

- *k-partition : family of k open, connected and disjoint subsets of M, $\mathcal{D} = \{D_1, \ldots, D_k\}$.*
- *Energy : $\Lambda_k(\mathcal{D}) = \max_{1 \leq i \leq k} \lambda_1(D_i)$.*
- *Minimal energy : $\mathfrak{L}_k(M) := \inf_{\mathcal{D}} \Lambda_k(\mathcal{D})$.*
- *Minimal k-partition $\mathcal{D}^*$: $\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(M)$.*

The max-min principle tells us that the nodal partition associated with a Courant-sharp eigenfunction is minimal.
For $n = 2$, the converse is true : any eigenfunction, whose associated nodal partition is minimal, is necessarily Courant-sharp (Helffer–Hoffmann-Ostenhof–Terracini, 2009).

Refinement of Courant theorem

We define $\nu_k = \max\{\nu(u)\,;\, u \in E_{\lambda_k(M)}\}$.

Theorem (Pleijel, 1956)

If Ω is a bounded open set in $\mathbb{R}^2$ with a regular boundary, only a finite number of eigenvalues are Courant-sharp. In fact

$$\limsup_{k\to\infty} \frac{\nu_k}{k} \leq \frac{4}{\lambda_1(\mathbb{D})} = \frac{4}{j_{0,1}^2} < 1.$$

Theorem (Bérard–Meyer, 1982)

For all $n \geq 2$, there exists $\gamma_n < 1$ such that, for all compact manifold M of dimension n,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma_n.$$

Here we are imposing a Dirichlet boundary condition on ∂M if $\partial M \neq \emptyset$.

Proof of Pleijel's result

- Let u, associated with $\lambda_k(\Omega)$, have ν_k nodal domains $D_1, \ldots, D_{\nu_k}$.
- Applying Faber-Krahn : $\lambda_k(\Omega) = \lambda_1(D_i) \geq \frac{\pi j_{0,1}^2}{|D_i|}$ pour $1 \leq i \leq \nu_k$.
- Summing : $\nu_k \pi j_{0,1}^2 \leq \lambda_k(\Omega)|\Omega|$.
- Weyl's law : $\lambda_k(\Omega) \sim \frac{4\pi k}{|\Omega|}$.
- Conclusion : $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2}$.

Additional remarks

- Asymptotic isoperimetric inequality for domains of small volume in the proof by Bérard–Meyer : for all $\varepsilon > 0$, there exists $V(\varepsilon, M) > 0$ such that, if $|D| \leq V(\varepsilon, M)$, $|\partial D||D|^{-\frac{n-1}{n}} \geq (1-\varepsilon)|\partial \mathbb{B}^n||\mathbb{B}^n|^{-\frac{n-1}{n}}$.
- Asymptotic Faber-Krahn inequality : if $|D| \leq V(\varepsilon, M)$, $\lambda_1(D)|D|^{\frac{2}{n}} \geq (1-\varepsilon)^2 \lambda_1(\mathbb{B}^n)|\mathbb{B}^n|^{\frac{2}{n}}$.
- The constant γ_n is explicit :

$$\gamma_n = \frac{(2\pi)^n}{\omega_n^2 \lambda_1(\mathbb{B}^n)^{n/2}} = \frac{(2\pi)^n}{\omega_n^2 j^n_{(n-2)/2,1}} = \frac{2^{n-2} n^2 \Gamma(\frac{n}{2})^2}{j^n_{(n-2)/2,1}} < 1.$$

  The sequence $(\gamma_n)_{n\geq 2}$ is decreasing and $\gamma_n = O\left(n\left(\frac{2}{e}\right)^n\right)$.
- Problem open for the Neumann boundary condition. Conjecture : same theorem, with the same constant. Proved in the case of a bounded open set in $\mathbb{R}^2$ with a piecewise analytic boundary (Polterovich, 2009). Key point : the number of nodal domain touching the boundary is controlled by $\sqrt{\lambda}$ (Toth–Zeldtich, 2009).

- Conjecture by Polterovich, in the case of a bounded domain in $\Omega \in \mathbb{R}^2$:

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{\lambda_1\left((0,\sqrt{\pi})^2\right)} = \frac{2}{\pi} \simeq 0.6366.$$

Compare with the upper bound $\frac{4}{j_{0,1}^2} \simeq 0.6917$.

- In the case of a rectangle $\mathcal{R}_{a,b} = (0,a) \times (0,b)$, with $\frac{b^2}{a^2} \notin \mathbb{Q}$, the set of limit points of the sequence $\left(\frac{\nu_k}{k}\right)_{k\geq 1}$ is the interval $\left[0, \frac{2}{\pi}\right]$.
- Corresponding conjecture in dimension $n > 2$:

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\omega_n^2 \lambda_1\left(\left(0, \omega_n^{1/n}\right)\right)^{n/2}} = \frac{2^n}{\omega_n n^{n/2}}.$$

- In the case of an irrational rectangular domain $\mathcal{R}_a = \Pi_{i=1}^n (0, a_i)$ (the a_i^{-2}'s are linearly independent over $\mathbb{Q}$), the set of limit points of $\left(\frac{\nu_k}{k}\right)_{k\geq 1}$ is the interval $\left[0, \frac{2^n}{\omega_n n^{n/2}}\right]$.

Some examples

All Courant-sharp eigenvalues besides $\lambda_1(M)$ and $\lambda_2(M)$ are known in some specific examples.

- Square, Dirichlet case (λ_4) (Pleijel, 1956 ; Bérard–Helffer 2014) ;
- Sphere $\mathbb{S}^2$ (none) (Leydold, 1996) ;
- Disk, Dirichlet case (λ_4) (Helffer–Hoffmann-Ostenhof–Terracini, 2009) ;
- Square, Neumann case (λ_4, λ_5, and λ_9) (Helffer–Persson-Sundqvist, 2014) ;
- Square and cubical tori (none) (L., 2014, L. 2015) ;
- Equilateral torus (none), equilateral (λ_4), hemi-equilateral (none) and right-angled isosceles (none) triangles (Bérard–Helffer, 2015) ;
- Disk, Neumann case (λ_4), $\mathbb{S}^{n-1}$ for $n \geq 4$ (none), and unit ball in dimension $n \geq 3$, Dirichlet and Neumann cases (none) (Helffer–Persson-Sundqvist, 2015) ;
- Cube, Dirichlet case (none) (Helffer–Kiwan, 2015) ;
- Right-angled isosceles triangle, Neumann case (λ_3, λ_4 and λ_6) (Band–Bersudsky–Fajman, 2015)

Plan

Presentation of the example

$\mathbb{T}^2$ flat square torus of dimension 2 : $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.

Eigenvalues of $-\Delta_{\mathbb{T}^2}$: $\lambda_{m,n} = 4\pi^2(m^2+n^2)$.

Eigenfunctions :

$$u^{cc}_{m,n}(x,y) = \cos(2m\pi x)\cos(2n\pi y);$$
$$u^{cs}_{m,n}(x,y) = \cos(2m\pi x)\sin(2n\pi y);$$
$$u^{sc}_{m,n}(x,y) = \sin(2m\pi x)\cos(2n\pi y);$$
$$u^{ss}_{m,n}(x,y) = \sin(2m\pi x)\sin(2n\pi y).$$

Vector space of eigenfunctions $E_{m,n}$, of dimension 1, 2 or 4.

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n)\in\mathbb{N}^2} E_{m,n}}.$$

$\lambda_1(\mathbb{T}^2) = \lambda_{0,0} = 0$ and $\lambda_k(\mathbb{T}^2) = \lambda_{1,0} = \lambda_{0,1} = 4\pi^2$ for $k \in \{2,3,4,5\}$.

Statement of the result

Theorem

The only Courant-sharp eigenfunctions of $-\Delta_{\mathbb{T}^2}$ are associated with $\lambda_k(\mathbb{T}^2)$ for $k \in \{1, 2, 3, 4, 5\}$ (first and second eigenvalues).

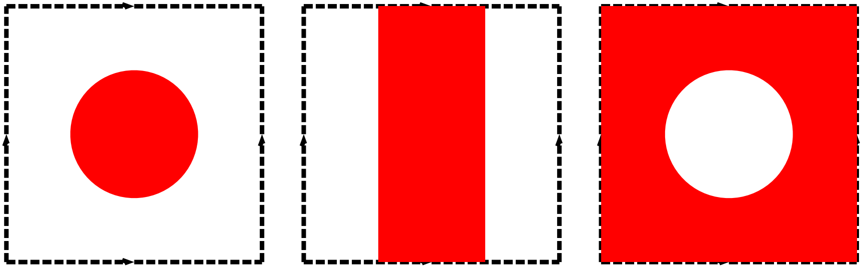
Corollary

The minimal k-partitions of $\mathbb{T}^2$ *are nodal only for* $k \in \{1, 2\}$.

Corollary

For $k \geq 3$, *we have* $\nu_k \leq k - 1$.

Isoperimetric domains for $\mathbb{T}^2$ (Howards–Hutchings–Morgan, 1999)

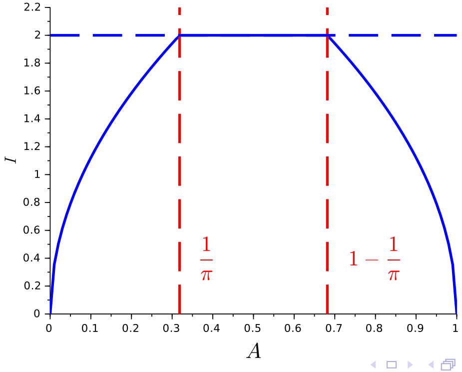


(d) $0 < A \leq \frac{1}{\pi}$ (e) $\frac{1}{\pi} \leq A \leq 1 - \frac{1}{\pi}$ (f) $1 - \frac{1}{\pi} \leq A < 1$

Isoperimetric profile for $\mathbb{T}^2$

For $A \in (0,1]$, $I(A) := \inf\left\{|\partial\Omega| \,:\, \Omega \subset \mathbb{T}^2 \text{ and } |\Omega| = A\right\}$.

$$I(A) = \begin{cases} 2(\pi A)^{1/2} & \text{if } 0 < A \leq \frac{1}{\pi} \quad \text{(disk);} \\ 2 & \text{if } \frac{1}{\pi} \leq V \leq 1 - \frac{1}{\pi} \quad \text{(strip);} \\ 2(\pi(1-A))^{1/2} & \text{if } 1 - \frac{1}{\pi} \leq A \quad \text{(complement of a disk).} \end{cases}$$



Faber-Krahn inequality for $\mathbb{T}^2$

Proposition

Let $D \subset \mathbb{T}^2$ such that $|D| \leq \frac{1}{\pi}$. Then $\lambda_1(D)|D| \geq \pi j_{0,1}^2$.

Proof : we use the co-area formula and apply Schwartz symmetrization to the level sets $D_t = \{x\,;\,u(x) > t\}$, where u is a positive eigenfunction associated with $\lambda_1(D)$. This works since all the level sets satisfy $|D_t| \leq \frac{1}{\pi}$.

Upper and lower bounds

Lemma

If λ is a Courant-sharp eigenvalue with $\kappa(\lambda) \geq 4$, then $\kappa(\lambda) \leq \frac{\lambda}{\pi j_{0,1}^2}$.

Proof : for u associated eigenfunction with $\kappa(\lambda)$ nodal domains, there is one nodal domain D satisfying $|D| \leq \frac{1}{\kappa(\lambda)} < \frac{1}{\pi}$, and therefore $\pi j_{0,1}^2 \leq \lambda_1(D)|D| \leq \frac{\lambda}{\kappa(\lambda)}$.

$N(\lambda) := \sharp\{k \,:\, \lambda_k(\mathbb{T}^2) < \lambda\}$ (counting function).

Lower bound : $N(\lambda) > \pi \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{\sqrt{2}}{2}\right)^2$.

For an eigenvalue λ, $\kappa(\lambda) = N(\lambda) + 1$.

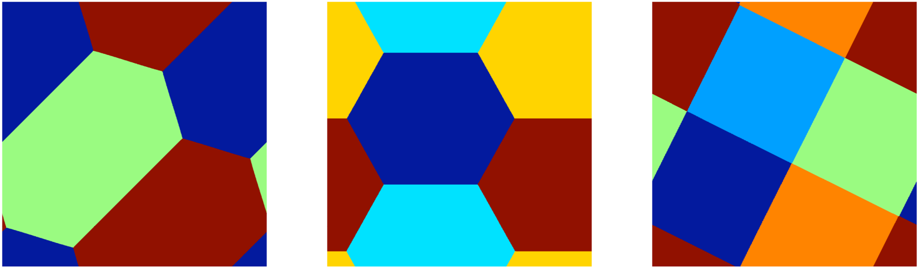
A priori bound : λ is not Courant-sharp if $\kappa(\lambda) \geq 27$.

Reduction to a finite list

$\frac{\lambda}{4\pi^2}$	indices	multiplicity	κ	$\frac{\lambda}{\pi j_{0,1}^2}$
0	$(0,0)$	1	1	
1	$(1,0)$, $(0,1)$	4	2	
2	$(1,1)$	4	6	4.35
4	$(2,0)$, $(0,2)$	4	10	8.69
5	$(2,1)$, $(1,2)$	8	14	10.86
8	$(2,2)$	4	22	17.38
9	$(3,0)$, $(0,3)$	4	26	19.56

TABLE : The 29 first eigenvalues

Numerical results



(g) $k = 3$ (h) $k = 4$ (i) $k = 5$

FIGURE : Minimal k-partitions of $\mathbb{T}^2$ for $k \in \{3, 4, 5\}$

For $k \in \{3, 4\}$, the tilings by hexagons are actually not minimal.

Plan

Presentation of the example

$\mathbb{T}^3$ the flat cubic torus of dimension 3 : $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$.

Eigenvalues of $-\Delta_{\mathbb{T}^3}$: $\lambda_{m,n,p} = 4\pi^2(m^2 + n^2 + p^2)$.

Eigenfunctions :

$$u_{m,n,p}(x,y,z) = \varphi(2m\pi x)\psi(2n\pi y)\chi(2p\pi z),$$

with φ, ψ, and χ in $\{\cos, \sin\}$.

Vector space of eigenfunctions $E_{m,n,p}$, of dimension 1, 2, 4 or 8.

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n,p)\in\mathbb{N}^2} E_{m,n,p}}.$$

$\lambda_1(\mathbb{T}^3) = \lambda_{0,0,0} = 0$ and $\lambda_k(\mathbb{T}^3) = \lambda_{1,0,0} = \lambda_{0,1,0} = \lambda_{0,0,1} = 4\pi^2$ for $k \in \{2,3,4,5,6,7\}$.

Statement of the result

Theorem

The only Courant-sharp eigenfunctions of $-\Delta_{\mathbb{T}^3}$ are associated with $\lambda_k(\mathbb{T}^3)$ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$ (first and second eigenvalues).

Corollary

For $k \geq 3$, *we have* $\nu_k \leq k - 1$.

Isoperimetric inequality

Main difficulty : the isoperimetric problem on the torus is not solved in dimension 3 or larger.

There are partial results for the periodic isoperimetric problems.

Theorem (Hauswirth–Perez–Romon–Ros, 2004)

Let $\mathcal{U} \subset \mathbb{T}^2 \times \mathbb{R}$ with $|\mathcal{U}| \leq \frac{4\pi}{81}$. Then

$$\left|\partial \mathbb{B}^3\right| \left|\mathbb{B}^3\right|^{-\frac{2}{3}} \leq |\partial \mathcal{U}| \, |\mathcal{U}|^{-\frac{2}{3}} .$$

Spheres-cylinders-planes profile

For $V \in (0, +\infty)$, $I(V) := \inf \left\{ |\partial\Omega| \,:\, \Omega \subset \mathbb{T}^2 \times \mathbb{R} \text{ and } |\Omega| = V \right\}$.

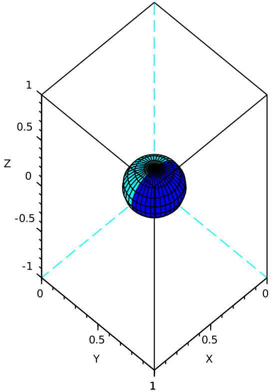
Minimizing among regions bounded by spheres, cylinders and pairs of two-dimensional planar tori produces the spheres-cylinders-planes profile . For $V \in (0, 1]$,

$$I_{SCP}(V) = \begin{cases} (36\pi)^{1/3} V^{2/3} & \text{if } 0 < V \leq \frac{4\pi}{81} & \text{(sphere);} \\ 2\pi^{1/2} V^{1/2} & \text{if } \frac{4\pi}{81} \leq V \leq \frac{1}{\pi} & \text{(cylinder);} \\ 2 & \text{if } \frac{1}{\pi} \leq V & \text{(pair of tori).} \end{cases}$$

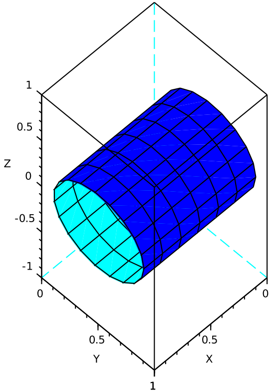
Conjecture : $I = I_{SCP}$.

Previous result : $I = I_{SCP}$ in the spherical range.

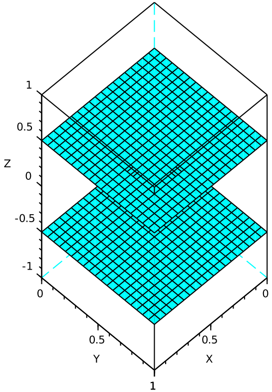
Conjectured isoperimetric domains for $\mathbb{T}^2 \times \mathbb{R}$



(a) $0 < V \leq \frac{4\pi}{81}$

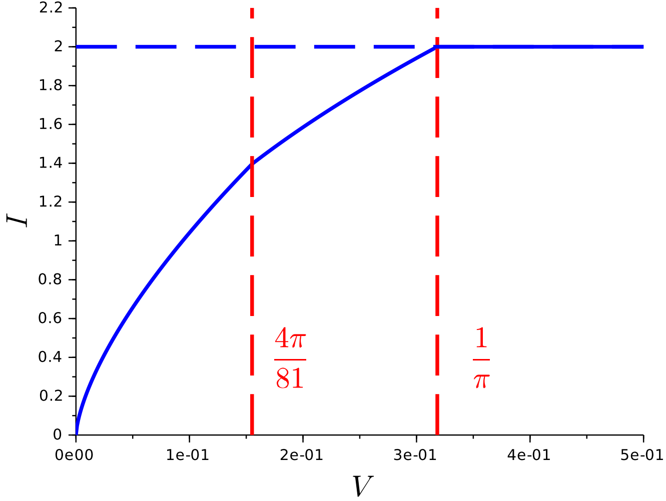


(b) $\frac{4\pi}{81} \leq V \leq \frac{1}{\pi}$



(c) $\frac{1}{\pi} \leq V$

Conjectured isoperimetric profile $\mathbb{T}^2 \times \mathbb{R}$



Inequalities in $\mathbb{T}^3$

Proposition

Let Ω be an open set in $\mathbb{T}^3$ with $|\Omega| \leq \frac{4\pi}{81}$. We have

$$\left|\partial\mathbb{B}^3\right| \left|\mathbb{B}^3\right|^{-\frac{2}{3}} \leq (|\partial\Omega| + 2\,|\Omega|)\,|\Omega|^{-\frac{2}{3}} .$$

Restatement : $\left(1 - \left(\frac{2|\Omega|}{9\pi}\right)^{\frac{1}{3}}\right) \left|\partial\mathbb{B}^3\right| \left|\mathbb{B}^3\right|^{-\frac{2}{3}} \leq |\partial\Omega|\,|\Omega|^{-\frac{2}{3}}$.

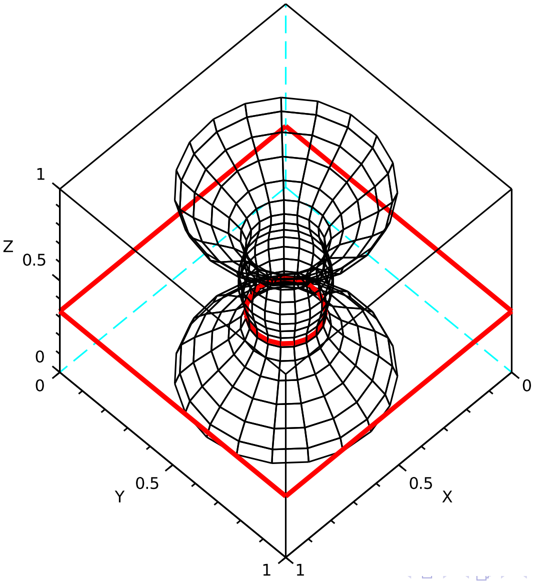
Corollary

Let Ω be an open set in $\mathbb{T}^3$ with $|\Omega| \leq \frac{4\pi}{81}$, we have

$$\left(1 - \left(\frac{2\,|\Omega|}{9\pi}\right)^{\frac{1}{3}}\right)^2 \lambda_1(\mathbb{B}^3) \left|\mathbb{B}^3\right|^{\frac{2}{3}} \leq \lambda_1(\Omega)\,|\Omega|^{\frac{2}{3}} .$$

Remark : $\lambda_1(\mathbb{B}^3) = \pi^2$.

Cutting procedure (adapted from Bérard–Meyer, 1982)



Proof of the isoperimetric inequality

Isoperimetric inequality :

- $\mathcal{H}_{z=t} := \{(x,y,z) \in \mathbb{T}^3 \,:\, z = t\}$.
- $|\Omega| = \int_0^1 |\Omega \cap \mathcal{H}_{z=t}| \, dt$.
- There exists $t_z \in (0,1)$ such that $|\Omega \cap \mathcal{H}_{z=t_z}| \leq |\Omega|$.
- We consider $\widetilde{\Omega} := \Omega \setminus \mathcal{H}_{z=t}$ as a subset of $\mathbb{T}^2 \times \mathbb{R}$; we have $|\widetilde{\Omega}| = |\Omega|$ and $|\partial\widetilde{\Omega}| \leq |\partial\Omega| + 2|\Omega|$.
- We apply the isoperimetric inequality in $\mathbb{T}^2 \times \mathbb{R}$.

Upper and lower bounds

Lemma

If λ is a Courant-sharp eigenvalue of $-\Delta_{\mathbb{T}^3}$ with $\kappa(\lambda) \geq 7$, then

$$\kappa(\lambda) \leq \left(\left(\frac{3}{4\pi^4} \right)^{\frac{1}{3}} \sqrt{\lambda} + \left(\frac{2}{9\pi} \right)^{\frac{1}{3}} \right)^3 .$$

Proposition

For $\lambda \geq 12\pi^2$, $N(\lambda) \geq \frac{4\pi}{3} \left(\frac{\sqrt{\lambda}}{2\pi} - \frac{\sqrt{3}}{2} \right)^3$.

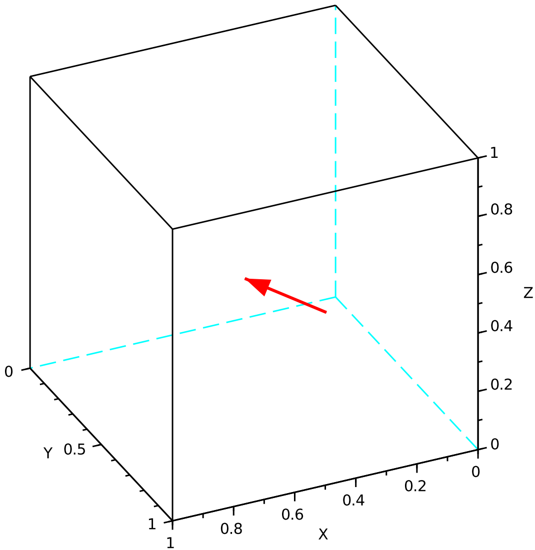
Reduction to a finite list

A priori bound : λ is not Courant-sharp if $\kappa(\lambda) \geq 270$.

After examination of the remaining eigenvalues, the only ones which can be Courant-sharp are :

- $\lambda_1(\mathbb{T}^3) = \lambda_{0,0,0} = 0$;
- for $k \in \{2, \ldots, 7\}$, $\lambda_k(\mathbb{T}^3) = \lambda_{1,0,0} = \lambda_{0,1,0} = \lambda_{0,0,1} = 4\pi^2$;
- for $k \in \{8, \ldots, 19\}$, $\lambda_k(\mathbb{T}^3) = \lambda_{1,1,0} = \lambda_{1,0,1} = \lambda_{0,1,1} = 8\pi^2$.

An isometry of $\mathbb{T}^3$



Courant Theorem with symmetry

- Isometry : $\sigma(x,y,z) := (x+1/2, y+1/2, z+1/2)$.
- Space of symmetric functions : $\mathcal{S} := \{u \in L^2(\mathbb{T}^3) \,:\, u \circ \sigma = u\}$.
- $H_{\mathcal{S}}$ restriction of $-\Delta_{\mathbb{T}^3}$ to $\mathcal{S}$.
- $(\lambda_k^{\mathcal{S}})_{k\geq 1}$ spectrum of $H_{\mathcal{S}}$.
- If λ eigenvalue of $H_{\mathcal{S}}$, $\kappa_{\mathcal{S}}(\lambda) := \inf\{k \,:\, \lambda_k^{\mathcal{S}} = \lambda\}$

If u is an eigenfunction of $H_{\mathcal{S}}$, and D a nodal domain of u, then $\sigma(D)$ is also a nodal domain of u. Either $\sigma(D) = D$: the domain is symmetric, or $\sigma(D) \neq D$: $\{D, \sigma(D)\}$ is a pair of isometric domains. We denote by $\alpha(u)$ the number of symmetric domains and by $\beta(u)$ the number of pairs. We have
$\nu(u) = \alpha(u) + 2\beta(u)$.

Courant theorem with symmetry (Leydold, 1996 ; Helffer–Hoffmann-Ostenhof–Terracini, 2010)

If u is an eigenfunction of $H_{\mathcal{S}}$ associated with the eigenvalue λ,

$$\alpha(u) + \beta(u) \leq \kappa_{\mathcal{S}}(\lambda).$$

Conclusion of the proof

Remark :

$$u_{m,n,p}(x+1/2, y+1/2, z+1/2) = \varphi(2m\pi x + m\pi)\psi(2n\pi y + n\pi)\chi(2p\pi z + p\pi) = (-1)^{m+n+p}u_{m,n,p}(x,y,z) = (-1)^{m^2+n^2+p^2}u_{m,n,p}(x,y,z)$$

Consequence : if u is an eigenfunction associated with the eigenvalue $8\pi^2$, it is symmetric , and $\nu(u) = \alpha(u) + 2\beta(u) \leq 2(\alpha(u) + \beta(u))$.

On the other hand, $8\pi^2$ is an eigenvalue of H_S with $\kappa_S(8\pi^2) = 2$. According to Courant theorem with symmetry, $\alpha(u) + \beta(u) \leq 2$.

Therefore, $\nu(u) \leq 4$ (sharp bound) while $\kappa(8\pi^2) = 8$.

The eigenvalue $8\pi^2$ of $-\Delta_{\mathbb{T}^3}$ is not Courant-sharp.

Remarks and open questions

- Does the same result holds for $\mathbb{T}^n$, $n \geq 4$?
- Hope : find a good enough isoperimetric inequality to show that λ is not Courant-shap if $\kappa(\lambda) > 2$.
- Remark : if $\kappa(\lambda) > 2$, $\kappa(\lambda) \geq 2(n+1)$, so we work with nodal domains of volume no larger than $\frac{1}{2(n+1)}$.
- We can make n cuts by planes, and we obtain, for $|\Omega| \leq \frac{\omega_n}{2^n}$,

$$\left(1 - \left(\frac{2^n\,|\Omega|}{\omega_n}\right)^{\frac{1}{n}}\right)|\partial\mathbb{B}^n|\,|\mathbb{B}^n|^{-\frac{n-1}{n}} \leq |\partial\Omega|\,|\Omega|^{-\frac{n-1}{n}} \ ,$$

  but $\frac{\omega_n}{2^n} \sim \frac{1}{\sqrt{\pi n}}\left(\frac{\pi e}{2n}\right)^{\frac{n}{2}}$.
- Even assuming cylinders are optimal, we recover the euclidean isoperimetric inequality for $|\Omega| \leq V_n$, with

$$V_n = \frac{(n-1)^{n(n-1)}\omega_{n-1}^n}{n^{n(n-1)}\omega_n^{n-1}} \sim \frac{e^{\frac{1}{4}-\frac{n}{2}}}{\sqrt{\pi n}}.$$

- Other types of isoperimetric inequalities, valid for all volumes ? The gaussian isoperimetric profile is inefficient for small volumes.

Plan

Parity of the nodal count

$$\mathbb{T}_b^2 := (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$$

Eigenvalues of $-\Delta_{\mathbb{T}_b^2}$:

$$\lambda_{m,n} = 4\pi^2 \left(m^2 + \frac{n^2}{b^2} \right) ,$$

Eigenfunctions $u^{cc}_{m,n}$, $u^{cs}_{m,n}$, $u^{sc}_{m,n}$, $u^{ss}_{m,n}$, with the same notation as before.

T. Hoffmann-Ostenhof. Geometric aspects of spectral theory (July 1st – July 7th, 2012), Problem Section (xv).
Oberwolfach Rep., 9(3) :2013–2076, 2012.T. Hoffmann-Ostenhof.

Problem

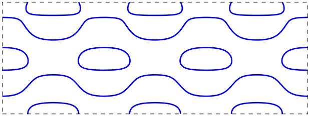
Is there a flat torus so that for some eigenfunction in the eigenspace $E_{\lambda_{m,n}}$ for some $\lambda_{m,n}$ there is an non-constant eigenfunction u with an odd number of nodal domains ?

Irrational case

Proposition

If b^2 is irrational, any non-constant eigenfunction u of $-\Delta_{\mathbb{T}^2_b}$ has an even number of nodal domains.

Nodal lines of $u^{cc}_{3,2} + \frac{1}{2} u^{ss}_{3,2}$ for $b = e^{-1}$:



Proof :

- If $m > 0$, set $\mathbf{v} := (1/2m, 0)$.
- For each basis function $u \in \{u^{cc}_{m,n}, u^{cs}_{m,n}, u^{sc}_{m,n}, u^{ss}_{m,n}\}$, we have $u(\mathbf{x} + \mathbf{v}) = -u(\mathbf{x})$.
  Example : $\cos(2\pi mx + \pi)\cos\left(\frac{2\pi ny}{b}\right) = -\cos(2\pi mx)\cos\left(\frac{2\pi ny}{b}\right)$
- We have a bijection between positive and negative nodal domains.
- If $m = 0$, $n > 0$, and we do the same with $\mathbf{v} := (0, b/2n)$.

The case $b = 1$

Proposition

If $b = 1$, any non-constant eigenfunction u of $-\Delta_{\mathbb{T}^2_b}$ has an even number of nodal domains.

In that case we can have higher multiplicities with pairs $(m, n) \neq (m', n')$ such that $m^2 + n^2 = m'^2 + n'^2$.

Lemma (Hoffmann-Ostenhof, 2015)

For $(m, n) \neq (0, 0)$, we write $\lambda = m^2 + n^2$. If $\lambda = 2^{2p}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $m = 2^p m_0$ and $n = 2^p n_0$, where exactly one of the integers m_0 and n_0 is odd. If $\lambda = 2^{2p+1}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $n = 2^p m_0$ and $n = 2^p n_0$, where both integers m_0 and n_0 are odd.

- In the first case, we set $\mathbf{v} := (1/2^{p+1}, 1/2^{p+1})$.
- In the second case, we set $\mathbf{v} := (1/2^{p+1}, 0)$ or $\mathbf{v} := (0, 1/2^{p+1})$.

Corollary

For $k \geq 4$, we have $\nu_k \leq k - 2$.

Counter-example for $b = \frac{1}{\sqrt{3}}$

Proposition

If $b = \frac{1}{\sqrt{3}}$, there exists an eigenfunction of $-\Delta_{\mathbb{T}^2_b}$ with three nodal domains.

Idea : consider $v_\varepsilon = u^{cc}_{1,1} + \varepsilon u^{cc}_{2,0}$.

(d) $\mathcal{N}(u^{cc}_{1,1})$ (e) $\mathcal{N}(u^{cc}_{2,0})$

FIGURE : Nodal sets of basis functions

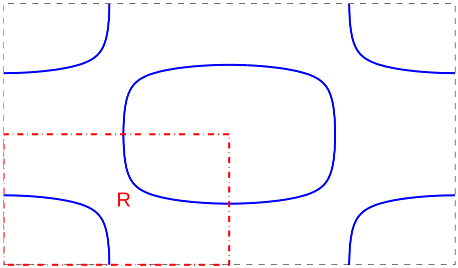


FIGURE : Nodal set of $v_\varepsilon = u^{cc}_{1,1} + \varepsilon u^{cc}_{2,0}$ with $\varepsilon = 0.1$

Proof

$$R := \left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2\sqrt{3}}\right[.$$

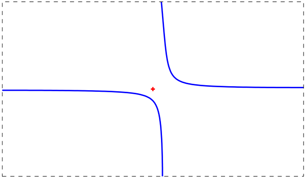
Change of coordinates :

$$\begin{cases} s = -\cos(2\pi x) ; \\ t = -\cos\left(\frac{2\pi y}{b}\right) . \end{cases}$$

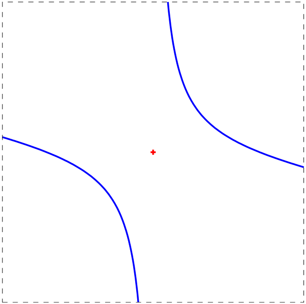
R is sent to $]-1,1[\times]-1,1[$. Nodal set of v_ε in the new coordinates :

$$st + \varepsilon(2s^2 - 1) = 0 ,$$

two branches of hyperbola.



(a) In (x, y)-coordinates



(b) In (u, v)-coordinates

FIGURE : Nodal set in R

Remarks

- The arithmetical lemma still holds for the equation $\lambda = \alpha m^2 + \beta n^2$, with $\alpha + \beta = 2 \bmod 4$. Therefore, if $b = \sqrt{\frac{\alpha}{\beta}}$ there can only be an even number of nodal domains.
- Assuming that
$$m^2 + \frac{n^2}{b^2} = k^2 m^2,$$
that is to say
$$b = \frac{n}{m\sqrt{k^2 - 1}},$$
$$v_\varepsilon = u^{cc}_{m,n} + \varepsilon u^{c,c}_{km,0}$$
is an eigenfunction, with $2mn + 1$ nodal domains for a small ε.
- Problem : characterizing the rational numbers q, such that there exists an eigenfunction on the torus $\mathbb{T}^2_{\sqrt{q}}$ with an odd number of nodal domains.