Nodal patterns of the Laplacian on thin domains

Asymptotic analysis and spectral theory

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Courant theorem

- Ω bounded open set in $\mathbb{R}^d$;
- $\lambda_1(\Omega) < \lambda_2(\Omega) \leq \dots \leq \lambda_k(\Omega) \leq \dots$ sequence of the eigenvalues of the Dirichlet-Laplacian $-\Delta_\Omega^D$, repeated according to their multiplicities.

If u eigenfunction :

- nodal set : $\mathcal{N}(u) := \overline{\{x \in \Omega\,;\, u(x) = 0\}}$;
- nodal domain : connected component of $\Omega \setminus \mathcal{N}(u)$;
- $\nu(u)$ number of nodal domains.

If λ eigenvalue, $\kappa(\lambda) := \min\{k \geq 1\,;\, \lambda = \lambda_k(\Omega)\}$.

Courant theorem

If u is an eigenfunction of $-\Delta_\Omega^D$ associated with the eigenvalue λ,

$$\nu(u) \leq \kappa(\lambda).$$

Pleijel theorem

Definition

The eigenvalue λ is called Courant-sharp if there exists an associated eigenfunction u such that $\nu(u) = \kappa(\lambda)$.

Remark : $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ are always Courant-sharp.

Theorem (Å. Pleijel, 1956)

If Ω is a bounded open set in $\mathbb{R}^2$ with a sufficiently regular boundary, only a finite number of the eigenvalues $(\lambda_k(\Omega))_{k\geq 1}$ are Courant-sharp.

Set $\nu_k := \max\{\nu(u)\,;\, u \text{ associated with } \lambda_k(\Omega)\}$. Then $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2} < 1$.

This is in contrast with the one-dimensional situation (Strum-Liouville problems) where the k-th eigenvalue is simple and the corresponding eigenfunctions have k nodal domains.

Application : minimal partitions

- k-partition : family of k open, connected and disjoint subsets in Ω, $\mathcal{D} = \{D_1, \dots, D_k\}$.
- Energy : $\Lambda_k(\mathcal{D}) := \max_{1 \leq i \leq k} \lambda_1(D_i)$.
- $\mathfrak{L}_k(\Omega) := \inf_{\mathcal{D}} \Lambda_k(\mathcal{D})$.
- Minimal k-partition $\mathcal{D}^*$: $\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(\Omega)$ (Existence and regularity : D. Bucur, G. Buttazzo, A. Henrot, 1998 ; M. Conti, S. Terracini, G. Verzini, 2005 ; L. A. Caffarelli, F. H. Lin, 2007 ; B. Helffer, T. Hoffmann-Ostenhof, S.Terracini, 2009).
- Nodal partition : family of the nodal domains of an eigenfunction of $-\Delta_\Omega^D$.
- A nodal partition associated with a Courant-sharp eigenvalue is minimal.
- The converse is true when $d = 2$: if the partition associated with (λ, u) is minimal then $\nu(u) = \kappa(\lambda)$ (B. Helffer, T. Hoffmann-Ostenhof, S. Terracini, 2009).
- Pleijel theorem : for a given domain in $\mathbb{R}^2$, minimal k-partitions are not nodal, except for of finite number of k's.

Explicit computations : rectangles

- $\mathcal{R}_\varepsilon = \{(x,y) \in \mathbb{R}^2 \,:\, 0 < x < \pi \text{ and } 0 < y < \varepsilon\pi\}$;
- $\lambda_{m,n}(\varepsilon) = m^2 + \frac{n^2}{\varepsilon^2}$;
- $u^\varepsilon_{m,n}(x,y) = \sin(mx)\sin\left(\frac{ny}{\varepsilon}\right)$;

If

$$k^2 + \frac{1}{\varepsilon^2} < 1 + \frac{4}{\varepsilon^2}, \text{ i.e. } \varepsilon < \sqrt{\frac{3}{k^2-1}},$$

the k first eigenvalues of $-\Delta^D_{\mathcal{R}_\varepsilon}$ are

$$\lambda_{1,1}(\varepsilon) < \lambda_{2,1}(\varepsilon) < \cdots < \lambda_{k,1}(\varepsilon),$$

$\lambda_{\ell,1}(\varepsilon)$ being simple and the associated eigenfunctions having ℓ nodal domains for $\ell \in \{1,\ldots,k\}$ (partition into ℓ equal vertical strips).
Basis for the numerical study of minimal partitions for the rectangle
(V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, B. Helffer, 2009).

Explicit computations : sectors I

- $\Sigma_\varepsilon = \{(\rho\cos(\theta), \rho\sin(\theta)) \mid 0 < \rho < 1 \text{ et } 0 < \theta < \varepsilon\}$;
- $\lambda_{m,n}(\varepsilon) = (j_{\frac{n\pi}{\varepsilon},m})^2$;
- $u^\varepsilon_{m,n}(\rho,\theta) = J_{\frac{n\pi}{\varepsilon}}(j_{\frac{n\pi}{\varepsilon},m}\,\rho)\sin\left(\frac{n\pi\theta}{\varepsilon}\right)$.

Here, $J_{\frac{n\pi}{\varepsilon}}$ is the Bessel function of the first kind with parameter $\frac{n\pi}{\varepsilon}$ and $j_{\frac{n\pi}{\varepsilon},m}$ is its m-th positive zero.
The function $u^\varepsilon_{m,n}$ has mn nodal domains.
We have the asymptotic expansion

$$\lambda_{m,n}(\varepsilon) = \frac{n^2\pi^2}{\varepsilon^2} + \frac{(2n^2\pi^2)^{2/3}}{\varepsilon^{4/3}}|a_m| + O\left(\varepsilon^{-2/3}\right),$$

where the a_m's are the zeros of the Airy function Ai, arranged in the following order

$$\cdots < a_m < \cdots < a_2 < a_1 < 0.$$

Explicit computations : sectors II

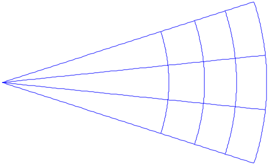


FIGURE : Domaines nodaux de $u_{4,3}^{\varepsilon}$ pour $\varepsilon = \frac{\pi}{5}$.

From this we conclude that for any integer k, there exists $\varepsilon_k > 0$ such that for $\varepsilon \in (0, \varepsilon_k)$, the k first eigenvalues of $-\Delta_{\Sigma_\varepsilon}^{D}$ are

$$\lambda_{1,1}(\varepsilon) < \lambda_{2,1}(\varepsilon) < \cdots < \lambda_{k,1}(\varepsilon),$$

$\lambda_{\ell,1}(\varepsilon)$ being simple and the associated eigenfunctions having ℓ nodal domains for $\ell \in \{1, \dots, k\}$ (partition by concentric arcs).
Basis for the numerical study of the minimal partitions (V. Bonnaillie-Noël, C. L., 2014).

Motivation for an asymptotic study

- The eigenvalues and eigenfunctions of the Laplacian on a ellipse should be computable (Mathieu functions). It is however difficult to extract the asymptotic information for thin domains.
- Can we generalize the observation that we are in a Courant-sharp situation for thin domains ?

Asymptotics of eigenvalues for thin domains I

Results by L. Friedlander and M. Solomyak, 2009 (using estimates on the resolvent) :

- Domains $\Omega_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, -a < x_1 < b \text{ and } 0 < x_2 < \varepsilon h(x_1)\}$ with $a, b > 0$
- We assume that h is positive and continuous on $I := [-a, b]$, belongs to $C^1(I \setminus \{0\})$, has a unique global maximum at 0, and satisfies the asymptotic expansions

$$h(x_1) = \begin{cases} M - c_+ x_1^m + O(x_1^{m+1}) & \text{for} \quad x_1 > 0, \\ M - c_- |x_1|^m + O(|x_1|^{m+1}) & \text{for} \quad x_1 < 0; \end{cases}$$

  with M, c_+, c_-, m positive real numbers, $m \geq 1$.
- We write $\lambda_k(\varepsilon) := \lambda_k(\Omega_\varepsilon)$.
- $\lambda_k(\varepsilon) = \frac{\pi^2}{M^2 \varepsilon^2} + \mu_k \varepsilon^{2\alpha} + o(\varepsilon^{2\alpha})$, with $\alpha = \frac{2}{m+2}$ and $(\mu_k)_{k\geq 1}$ the eigenvalues of the operator $-\frac{d^2}{dt^2} + V(t)$ in $L^2(\mathbb{R})$, with

$$V(t) = \begin{cases} \frac{2\pi}{M^3} c_+ t^m & \text{for} \quad t > 0, \\ \frac{2\pi}{M^3} c_- |t|^m & \text{for} \quad t < 0. \end{cases}$$

Asymptotics of eigenvalues for thin domains II

Results by D. Borisov and P. Freitas, 2007 (extension to domains in $\mathbb{R}^d$, 2009) :

- Domains
  $\Omega_\varepsilon := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < 1 \text{ and } -\varepsilon h_-(x_1) < x_2 < \varepsilon h_+(x_1)\}$.
- We assume that h_+ and h_- are continuous on $[0,1]$, that $H = h_+ + h_-$ is a positive function with a unique global maximum at some point $\overline{x} \in (0,1)$, that H is C^∞ in a neighbourhood of $\overline{x}$ and has the Taylor expansion

  $$H(x_1) \sim H_0 + \sum_{i=2p}^{\infty} \frac{H_i}{i!}(x_1 - \overline{x})^i,$$

  with $H_{2p} < 0$.
- For all $m, n \in \{1, 2, 3, \dots\}$ there exist eigenvalues of the $-\Delta^D_{\Omega_\varepsilon}$ with a complete asymptotic expansion of the form

  $$\lambda_{m,n}(\varepsilon) \sim c_0^{m,n}\varepsilon^{-2} + \varepsilon^{-2}\sum_{i=2p}^{\infty} c_i^{m,n}\varepsilon^{\frac{i}{p+1}}.$$

Asymptotics of eigenvalues for thin domains III

Results by D. Borisov and P. Freitas, 2007 (cont.) :

- The coefficients $c_i^{m,n}$ can be expressed using the H_i's and the eigenvalues $(\Lambda_{m,n})_{m\geq 1}$, and associated orthonormalized eigenfunctions , of the operators in $L^2(\mathbb{R})$,

$$G_n := -\frac{d^2}{dt^2} - \frac{2\pi^2 n^2 H_{2p}}{(2p)! H_0^3} t^{2p}.$$

- Explicitly, $c_0^{m,n} = \frac{\pi^2 n^2}{H_0^2}$, : $c_{2p}^{m,n} = \Lambda_{m,n}$, $c_{2p+1}^{m,n} = 0$, ...
- Furthermore, for all integer $k \geq 1$, there exists ε_k such that, for all $\varepsilon \in (0, \varepsilon_k)$, the k first eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_k(\varepsilon)$ are simple and correspond to $\lambda_{1,1}(\varepsilon), \dots, \lambda_{k,1}(\varepsilon)$.
- For an ellipse centered at $(1/2, 0)$ and having semi-axes $1/2$ and $\varepsilon/2$, we obtain

$$\lambda_1(\varepsilon) = \frac{\pi^2}{\varepsilon^2} + \frac{2\pi}{\varepsilon} + 3 + \left(\frac{11}{2\pi} + \frac{\pi}{3}\right)\varepsilon + O(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

Asymptotics of eigenvalues for thin domains IV

Study of triangles and cones in the small angle limits.

- P. Freitas, 2007.
  Four terms asymptotic expansion for the first eigenvalue of thin triangles (including isoceles and right angled triangles).
- M. Dauge, N. Raymond, 2012.
  Thin isosceles triangles, complete asymptotic expansion of the eigenvalues into powers of $\tan(\alpha)^{1/3}$ (α angular opening).
- T. Ourmière-Bonafos, 2014.
  Complete asymptotic expansions of the eigenvalues for general triangles, and for cones and spherical cones. Tunnelling.

Asymtotics of nodal sets for thin domains I

The Nodal Line Conjoncture : the nodal set of a second eigenfunction of the Dirichlet-Laplacian on a domain touches the boundary.

- Proved in the case of convex planar domains (A. D. Melas, 1991 ; G. Alessandrini, 1992).
- Disproved in the case of non-simply connected planar domains (M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, 1997).

Results by D. Jerison 1991, D. Grieser and D. Jerison, 1995.

- If the ratio of the inradius to the diameter of a convex planar domain is small enough, the nodal set of a second eigenfunction touches the boundary (extended to higher dimension (Jerison, 1995)).
- The first nodal line is 'almost a straight line' : if we rotate the domain so that the projection on $\{x_1 = 0\}$ has minimal length, and we rescale so that the minimal length is ε and the length in the orthogonal direction is 1, then the nodal set of a second eigenfunction is contained in a set of the form
$$\{|x_1 - a| < C\varepsilon^2\}$$
for some a, with C a universal constant.

Asymptotics of nodal sets for thin domains II

Results by P. Freitas and D. Krejčiřík, 2007

- With a regular curve $\Gamma : I \to \mathbb{R}^d$ (parametrized by arc length), a bounded open set $\omega \subset \mathbb{R}^{d-1}$, and a parameter $\varepsilon > 0$ small enough, we associate a 'tube' Ω_ε in $\mathbb{R}^d$ with 'constant cross-section' ω, shrinking to Γ. A point $x \in \Omega_\varepsilon$ is specified by its coordinates $(s, y) \in I \times \omega$.
- We consider the differential operator in $L^2(I)$:

$$-\frac{d^2}{ds^2} - \frac{\kappa_1(s)^2}{4}$$

  with Dirichlet boundary conditions. Here κ_1 is the first curvature of Γ. We denote by $(\mu_k)_{k\geq 1}$ its eigenvalues, by $(\varphi_k)_{k\geq 1}$ a corresponding orthonormal basis of eigenfunctions, and by $(s_{k,\ell})_{1\leq \ell \leq k-1}$ the zeros of φ_k.
- u_1 positive normalized eigenfunction associated with $\lambda_1(\omega)$.
- We have $\lambda_k(\varepsilon) = \varepsilon^{-2}\lambda_1(\omega) + \mu_k + O(\varepsilon)$.

Asymptotics of nodal sets for thin domains III

Results by P. Freitas and D. Krejčiřík, 2007 (cont.)

- An eigenfunction associated with $\lambda_k(\varepsilon)$ is approximated by $\varphi_k(s)u_1(y)$.
- For any integer $k \geq 1$, there exist M_k and ε_k positive such that, for all $\varepsilon \in (0, \varepsilon_k)$,
  1. $\lambda_k(\varepsilon)$ is simple (and we denote by u_k an associated eigenfunction) ;
  2. $\mathcal{N}(u_k) \subset \bigcup_{\ell=1}^{k-1}\{(s,y) \in I \times \omega \ : \ s_{k,\ell} - M_k\varepsilon < s < s_{k,\ell} + M_k\varepsilon\}$;
  3. $\mathcal{N}(u_k) \cap \partial\Omega_\varepsilon \neq \emptyset$;
  4. u_k has exactly k nodal domains.

Asymptotics of nodal sets for thin domains IV

Results of J. Lampart, 2014

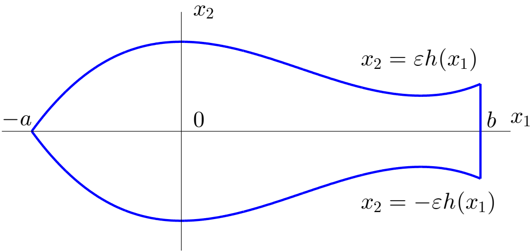
- General framework for adiabatic limits of the Laplacian on fibre bundles $\pi : G \to M$ equipped with Riemannian metrics g_ε such that the ratio of the diameter of the fibres to that of the base is given by ε ($\varepsilon \to 0$).
- Construction of an effective operator with an effective potential given by the eigenbands (eigenvalues of the fibre Laplacian).
- Approximation of eigenvalues and eigenfunctions for
  - constant ground-state ;
  - ground-state with a non-degenerate minimum.
- Localization of the nodal set in both cases for a base space B without boundary.

Plan

Thin domains

Let $h \in C^\infty((-a,b),\mathbb{R})$ $(a,b>0)$, with $h>0$, $h(0)$ unique maximum, $h''(0)<0$. For all $\varepsilon>0$, we write

$$\Omega_\varepsilon := \left\{(x_1,x_2)\in\mathbb{R}^2\,;\,-a<x_1<b \text{ and } -\varepsilon h(x_1)<x_2<\varepsilon h(x_1)\right\}.$$



Outline of the approximation I

We perform a Born-Oppenheimer approximation. The fast variable is x_2 and the slow variable x_1.

For the fast variable, we consider the differential operator

$$-\frac{d^2}{dx_2^2} \text{ on } (-\varepsilon h(x_1), \varepsilon h(x_1)),$$

with Dirichlet boundary conditions. Its first eigenvalue is

$$\frac{\pi^2}{4\varepsilon^2 h(x_1)^2}.$$

Multiplying by ε^2 and replacing the derivative with respect to x_2 by the first eigenvalue above, we obtain for the slow variable the differential operator

$$H_\varepsilon = -\varepsilon^2 \frac{d^2}{dx_1^2} + \frac{\pi^2}{4h(x_1)^2} \text{ on } (-a, b),$$

with Dirichlet boundary conditions.

Outline of the approximation II

The effective potential has an asymptotic expansion of the form

$$\frac{\pi^2}{4h(x_1)^2} = \frac{\pi^2}{4}\left(\kappa_0 + \kappa_2 x_1^2 + O(x_1^3)\right),$$

with κ_0 and κ_2 positive.
We perform a semi-classical approximation. The eigenvalues of H_ε admit the asymptotic expansions

$$\frac{\pi^2}{4}\kappa_0 + \nu_k \varepsilon + O(\varepsilon^{3/2})$$

with $(\nu_k)_{k\geq 1}$ eigenvalues of the harmonic oscillator

$$H = -\frac{d^2}{dt^2} + \frac{\pi^2 \kappa_2}{4} t^2.$$

Outline of the approximation III

Approximation for the eigenvalues of $-\Delta_\varepsilon^D$:

$$\lambda_k(\varepsilon) \simeq \frac{\pi^2}{4h(0)^2\varepsilon^2} + \left(k - \frac{1}{2}\right)\frac{\pi}{\varepsilon}\sqrt{\frac{|h''(0)|}{h(0)^3}}$$

Approximation for the corresponding eigenfunctions :

$$\varphi_k(\varepsilon^{-1/2}x_1)\cos\left(\frac{\pi x_2}{2\varepsilon h(x_1)}\right),$$

with

$$\varphi_k(x_1) = h_k\left(\left(\frac{|h''(0)|}{h(0)^3}\right)^{\frac{1}{4}}\sqrt{\pi}x_1\right)$$

where h_k is the k-th eigenfunction of $-\frac{d^2}{ds^2} + s^2$.

Statement of the results : eigenvalues

Proposition

Let us write $\mu_k(\varepsilon) = \varepsilon^2 \lambda_k(\varepsilon)$ *(eigenvalues of* $-\varepsilon^2 \Delta^D_{\Omega_\varepsilon}$*). For every positive integer* k *, there exists a sequence* $(\mu_{k,n})_{n \geq 0}$ *such that, for all integer* N *,*

$$\mu_k(\varepsilon) = \sum_{n=0}^{N} \mu_{k,n} \varepsilon^{n/2} + O(\varepsilon^{(N+1)/2})$$

as $\varepsilon \to 0$ *. Furthermore,*

$$\mu_{k,0} = \kappa_0 \frac{\pi^2}{4}, \; \mu_{k,1} = 0 \text{ and } \mu_{k,2} = \nu_k .$$

Statement of the results : eigenfunctions

We denote by $(u_{k,\varepsilon})_{k\geq 1}$ an orthonormal basis of eigenfunctions of $-\Delta^D_{\Omega_\varepsilon}$ associated with the sequence of eigenvalues $(\lambda_k(\varepsilon))_{k\geq 1}$.
We write $(a_{k,i})_{1\leq i\leq k-1}$ the zeros of φ_k (eigenfunction of S_2 associated with ν_k).

Proposition

For each positive integer k, there exists $\varepsilon_k > 0$ (small enough) and $M_k > 0$ (large enough) such that, for $\varepsilon \in (0, \varepsilon_k)$,

1. $\mathcal{N}(u_{k,\varepsilon}) \subset \bigcup_{\ell=1}^{k-1}\{(x_1, x_2) \in \Omega_\varepsilon \; : \; \sqrt{\varepsilon}a_{k,\ell} - M_k\varepsilon < x_1 \leq \sqrt{\varepsilon}a_{k,\ell} + M_k\varepsilon\}$,
2. $\mathcal{N}(u_{k,\varepsilon}) \cap \partial\Omega_\varepsilon \neq \emptyset$;
3. *$u_{k,\varepsilon}$ has k nodal domains.*

Plan

Operator in the new variables

We perform the change of variables

$$\Phi_\varepsilon : \begin{cases} y_1 & = & \varepsilon^{-1/2} x_1 \, ; \\ y_2 & = & \frac{x_2}{\varepsilon h(x_1)} \, . \end{cases}$$

We have

$$\Phi_\varepsilon(\Omega_\varepsilon) = (-\varepsilon^{-1/2} a, \varepsilon^{-1/2} b) \times (-1, 1),$$

and the differential operator $-\Delta$, written in the new variables, is

$$S(\varepsilon) = -\kappa(\varepsilon^{1/2} y_1) \partial^2_{y_2} - \varepsilon \left(\partial_{y_1} - \varepsilon^{1/2} y_2 \theta(\varepsilon^{1/2} y_1) \partial_{y_2} \right)^2$$

with

$$\kappa(t) = \frac{1}{h(t)^2} \text{ and } \theta(t) = \frac{h'(t)}{h(t)}.$$

Formal expansion of the operator

We write the Taylor expansions at 0 :

$$\kappa(t) \sim \kappa_0 + \sum_{i\geq 1} \kappa_i t^i \, (\kappa_2 > 0) \, , \theta(t) \sim \sum_{i\geq 1} \theta_i t^i , \text{ and } \theta(t)^2 \sim \sum_{i\geq 2} \gamma_i t^2 .$$

We obtain for $S(\varepsilon)$ the formal asymptotic expansion

$$S(\varepsilon) \sim \sum_{i\geq 0} \varepsilon^{i/2} S_i ,$$

with

$$S_0 = -\kappa_0 \partial_{y_2}^2 , \; S_1 = 0 \, , S_2 = -\partial_{y_1}^2 - \kappa_2 y_2^2 \partial_{y_2}^2 , \; S_3 = -\kappa_3 y_1^3 \partial_{y_2}^2$$

and, for $i \geq 4$,

$$S_i = 2y_1^{i-3} y_2 \theta_{i-3} \partial_{y_1} \partial_{y_2} - (\kappa_i y_1^i + y_1^{n-4} y_2^2 \gamma_{i-4}) \partial_{y_2}^2 + ((i-3)\theta_{i-3} - \gamma_{i-4}) y_1^{i-4} y_2 \partial_{y_2} ,$$

differential operators on the infinite strip

$$\Pi = \mathbb{R} \times (-1, 1).$$

Formal series

We define $\mathcal{H}$ as the space of finite linear combinations of functions of the form

$$(y_1, y_2) \mapsto f(y_1)g(y_2),$$

where $f \in \mathcal{S}(\mathbb{R})$ and $g \in C^\infty([-1,1])$ with $g(1) = g(-1) = 0$.

Proposition

For every positive integer k, there exist a sequence $(\mu_{k,n})_{n\geq 0}$ of real numbers and a sequence $(v_{k,n})_{n\geq 0}$ of function in $\mathcal{H}$ such that, for all $n \geq 0$,

$$\sum_{p+q=n} (S_p - \mu_{k,p})\, v_{k,q} = 0\,,$$

with

- $v_{k,0}$ *non-zero* ;
- $\mu_{k,0} = \kappa_0 \frac{\pi^2}{4}$;
- $\mu_{k,2} = \nu_k$, *k-th eigenvalue of the harmonic oscillator* $H = -\frac{d^2}{dx_1^2} + \frac{\kappa_2 \pi^2}{4} y_2^2$.

Construction of the quasi-modes

We fix a cut-off function χ such that

- $\chi \in C^\infty(\mathbb{R})$,
- $\forall t \in \mathbb{R},\ 0 \leq \chi(t) \leq 1$,
- $\text{supp}(\chi) \subset]-a/2, b/2[$,
- $\forall t \in]-a/4, b/4[,\ \chi(t) = 1$.

For every positive integer N, we write

$$\mu_{k,N}(\varepsilon) = \sum_{n=0}^{N} \mu_{k,n}\varepsilon^{n/2},$$

$$v_{k,\varepsilon,N}(y_1, y_2) = \chi(\varepsilon^{1/2} y_1) \sum_{n=0}^{N} \varepsilon^{n/2} v_{k,n}(y_1, y_2).$$

Pull-back and push-forward

- $\Pi_{\varepsilon,loc} = \left\{(y_1, y_2) \in \Pi\,;\, -\varepsilon^{-1/2}a/2 < y_1 < \varepsilon^{-1/2}b/2\right\}$;
- $\Omega_{\varepsilon,loc} = \left\{(x_1, x_2) \in \Omega_\varepsilon\,;\, -a/2 < x_1 < b/2\right\}$.
- We recall
$$\Phi_\varepsilon : \begin{cases} y_1 &= \varepsilon^{-1/2}x_1\,; \\ y_2 &= \frac{x_2}{\varepsilon h(x_1)}\,. \end{cases}$$
- We define
$$\begin{array}{rccc} (\Phi_\varepsilon)_* : & L^2(\Pi_{\varepsilon,loc}) & \to & L^2(\Omega_{\varepsilon,loc}) \\ & v & \mapsto & v \circ \Phi_\varepsilon \end{array}$$
and
$$\begin{array}{rccc} (\Phi_\varepsilon)^* : & L^2(\Omega_{\varepsilon,loc}) & \to & L^2(\Pi_{\varepsilon,loc}) \\ & v & \mapsto & v \circ \Phi_\varepsilon^{-1} \end{array}$$

Localization of eigenvalues

For every positive integer N we define $u_{k,\varepsilon,N} := (\Phi_\varepsilon)_*(v_{k,\varepsilon,N})$. We obtain that

$$\frac{\left\| \left(-\varepsilon^2 \Delta^D_{\Omega_\varepsilon} - \nu(\varepsilon)\right) u_\varepsilon \right\|_{L^2(\Omega_\varepsilon)}}{\left\| u_\varepsilon \right\|_{L^2(\Omega_\varepsilon)}} = O\left(\varepsilon^{(N+1)/2}\right).$$

The spectral theorem gives us the following.

Proposition

Let k and N be positive integers. There exist positive constants $\varepsilon_{k,N}$ and $C_{k,N}$ such that, for all $\varepsilon \in (0, \varepsilon_{k,N})$,

$$\left(\mu_{k,N}(\varepsilon) - C_{k,N}\,\varepsilon^{(N+1)/2}, \mu_{k,N}(\varepsilon) + C_{k,N}\,\varepsilon^{(N+1)/2}\right) \cap \sigma\left(-\Delta^D_{\Omega_\varepsilon}\right) \neq \emptyset.$$

Plan

Convergence to eigenvalues

$p_k(\varepsilon)$ orthogonal projector from $L^2(\Omega_\varepsilon)$ to the eigenspace of $\mu_k(\varepsilon)$.

Proposition

For all positive integers k and N,

$$\frac{\|u_{k,N,\varepsilon} - p_k(\varepsilon)u_{k,N,\varepsilon}\|_{L^2(\Omega_\varepsilon)}}{\|u_{k,N,\varepsilon}\|_{L^2(\Omega_\varepsilon)}} = O\left(\varepsilon^{(N-1)/2}\right) .$$

This follows from the fact that eigenvalues are separated by gaps of order ε.

Proposition

For all positive integers k and N,

$$\frac{\|\nabla\left(u_{k,N,\varepsilon} - p_k(\varepsilon)u_{k,N,\varepsilon}\right)\|_{L^2(\Omega_\varepsilon)}}{\|u_{k,N,\varepsilon}\|_{L^2(\Omega_\varepsilon)}} = O\left(\varepsilon^{(N-3)/2}\right) .$$

This follows from the variational formulation of the eigenvalue equation.

Projection on the eigenspaces and push-forward

We define

$$\chi : \begin{array}{ccc} L^2(\Omega_\varepsilon) & \to & L^2(\Omega_{\varepsilon,loc}) \\ u(x_1,x_2) & \mapsto & \chi(x_1)u(x_1,x_2). \end{array}$$

We write

$$w_{k,\varepsilon,N} = (\Phi_\varepsilon)_* \chi P_{k,\varepsilon} u_{k,\varepsilon,N},$$

extended by 0 to Π.

We want a uniform control of the approximation of $w_{k,N,\varepsilon}$ by $v_{k,\varepsilon,N}$ in order to localize the nodal set. We will use elliptic estimates.

Divergence form of the operator

The differential operator $S(\varepsilon)$ can be written in the form :

$$S(\varepsilon) = -\rho(\varepsilon^{1/2} y_1)\mathrm{div}\left(A(y,\varepsilon)\nabla\right),$$

with

$$A(y,\varepsilon) = \begin{pmatrix} \varepsilon^{1/2} & 0 \\ 0 & 1 \end{pmatrix} A_0(\varepsilon^{1/2} y_1, \varepsilon y_2) \begin{pmatrix} \varepsilon^{1/2} & 0 \\ 0 & 1 \end{pmatrix}$$

where $z_1 \mapsto \rho(z_1)$ and $z \mapsto A_0(z)$ is a C^∞-function, bounded as well as all of their derivatives.

Furthermore, there exists $c > 0$ such that

$$\forall z \in \Pi,\ \forall \xi \in \mathbb{R}^2,\ c\,\|\xi\|^2 \leq \xi^T A_0(z)\xi$$

(uniform ellipticity condition).

ε-norms and ε-regularity estimates

Definition

For $\varepsilon > 0$, we define the norm $\|v\|_{H^p_\varepsilon(\Pi)}$ of a function $v \in H^p(\Pi)$ by

- $\forall v \in L^2(\Pi)$, $\|v\|_{H^0_\varepsilon(\Pi)} = \|v\|_{L^2(\Pi)}$,
- $\forall p \in \mathbb{N}$, $\forall v \in H^{p+1}(\Pi)$,
  $\|v\|^2_{H^{p+1}_\varepsilon(\Pi)} = \varepsilon \|\partial_{y_1} v\|^2_{H^p_\varepsilon(\Pi)} + \|\partial_{y_2} v\|^2_{H^p_\varepsilon(\Pi)} + \|v\|^2_{H^p_\varepsilon(\Pi)}$.

Proposition

For all positive integer p, there exist constants $C_p \geq 0$ and $0 < \varepsilon_p \leq 1$ such that, for all $\varepsilon \in (0, \varepsilon_p)$, if $v \in H^1_0(\Pi)$, $f \in \bigcap_{\ell \geq 0} H^\ell(\Pi)$, and

$$S(\varepsilon)v = f,$$

we have

$$\|v\|_{H^{p+2}_\varepsilon(\Pi)} \leq C_p \|f\|_{H^p_\varepsilon(\Pi)}.$$

We write

$$\delta_{k,N,\varepsilon} = w_{k,N,\varepsilon} - v_{k,N,\varepsilon}.$$

By a bootstrap argument using the previous regularity theorem, we find (up to further restriction to a neighbourhood of $y_1 = 0$ of size $\varepsilon^{-1/2}$), that

$$\|\delta_{k,4,\varepsilon}\|_{H^3_\varepsilon} = O\left(\varepsilon^{3/2}\right).$$

We use the Sobolev embedding $H^2(\Pi) \subset L^\infty(\Pi)$ to obtain the following result.

Lemma

For all positive integer k, there exist $0 < \varepsilon_k \leq 1$ and $C_k \geq 0$ such that, for all $\varepsilon \in (0, \varepsilon_k)$,

$$\begin{aligned} \|\delta_{k,4,\varepsilon}\|_{L^\infty(\Pi)} &\leq C_k\,\varepsilon^{1/2}\,; \\ \|\partial_{y_1}\delta_{k,4,\varepsilon}\|_{L^\infty(\Pi)} &\leq C_k\,; \\ \|\partial_{y_2}\delta_{k,4,\varepsilon}\|_{L^\infty(\Pi)} &\leq C_k\,\varepsilon^{1/2}\,. \end{aligned}$$

Considering the form of $v_{k,4,\varepsilon}$, the lemma is also true for $\delta_{k,\varepsilon} = w_{k,4,\varepsilon} - v_{k,0}$.

We recall

$$v_{k,0}(y_1, y_2) = \varphi_k(y_1) \cos\left(\frac{\pi y_2}{2}\right)$$

We write $(a_{k,i})_{1 \le i \le k-1}$ the zeros of φ_k. We consider the subsets of Π defined by

$$E_{k,0,\varepsilon} = (a_0, a_{k,1} - M_k \varepsilon^{1/2}) \times (-1, 1),$$

$$E_{k,\ell,\varepsilon} = (a_{k,\ell} + M_k \varepsilon^{1/2}, a_{k,\ell+1} - M_k \varepsilon^{1/2}) \times (-1, 1) \text{ for } \ell \in \{1, \dots, k-2\},$$

and

$$E_{k,k,\varepsilon} = (a_{k,k-1} + M_k \varepsilon^{1/2}, a_k) \times (-1, 1),$$

with $a_0, a_k \in \mathbb{R}$ and M_k positive constant to be specified.

Lemma

We can chose M_k such that for ε small enough, $w_{k,4,\varepsilon}$ does not vanish in $E_{k,\ell}$.

Let us consider $(y_1, y_2) \in E_{k,\ell,\varepsilon}$. We have

$$|v_{k,0}(y_1, y_2)| \geq \inf_{t \in (a_{k,\ell} + C\varepsilon^{1/2}, a_{k,\ell+1} - C\varepsilon^{1/2})} |\varphi_k(t)| \mathrm{dist}(y_2, \{-1, 1\}).$$

On the other hand,

$$\begin{aligned} |v_{k,0}(y_1, y_2) - w_{k,4,\varepsilon}(y_1, y_2)| &\leq \|\partial_{y_2} \delta_{k,\varepsilon}\|_{L^\infty(\Pi)} \mathrm{dist}(y_2, \{-1, 1\}) \\ &\leq C_k \, \varepsilon^{1/2} \mathrm{dist}(y_2, \{-1, 1\}). \end{aligned}$$

Since $\varphi_k(a_{k,\ell}) \neq 0$ and $\varphi_k(a_{k,\ell+1}) \neq 0$, we can choose M_k large enough so that

$$\inf_{t \in (a_{k,\ell} + M_k\varepsilon^{1/2}, a_{k,\ell+1} - M_k\varepsilon^{1/2})} |\varphi_k(t)| \geq \widetilde{C}_k \varepsilon^{1/2}$$

with $\widetilde{C}_k > C_k$.
In that case, $w_{k,4,\varepsilon}$ has the sign of $v_{k,0}$ on $E_{k,\ell,\varepsilon}$ for ε small enough.
We conclude using Courant theorem.