On the number of nodal domains for flat tori

SMS 2015, Montreal

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24 June 2015

Plan

1. Review of general results
2. Courant-sharpness for the square torus
3. Parity of the number of nodal domains on rectangular tori

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Introduction

- M compact manifold (with or without boundary) of dimension n ;
- $\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \dots$ sequence of the eigenvalues of $-\Delta_M$, with Dirichlet condition, counted with multiplicities.

If u eigenfunction :

- nodal set : $\mathcal{N}(u) = \overline{\{x \in M\,;\, u(x) = 0\}}$;
- nodal domain : connected component of $M \setminus \mathcal{N}(u)$;
- $\nu(u)$ number of nodal domains.

If λ eigenvalue, $\kappa(\lambda) = \min\{k \geq 1\,;\, \lambda = \lambda_k(M)\}$.

Courant Theorem

If u is an eigenfunction of $-\Delta_M$ associated with the eigenvalue λ,

$$\nu(u) \leq \kappa(\lambda).$$

Refinements of Courant Theorem

Definition

The eigenvalue λ is called Courant-sharp if there exists an associated eigenfunction u such that $\nu(u) = \kappa(\lambda)$.

Theorem (Pleijel, 1956)

If Ω is a bounded open set in $\mathbb{R}^2$ with a regular boundary, only a finite number of eigenvalues $(\lambda_k(\Omega))_{k\geq 1}$ are Courant-sharp.

Set $\nu_k := \max\{\nu(u)\,;\, u \text{ associated with } \lambda_k(\Omega)\}$. Then $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2} < 1$.

Theorem (Bérard–Meyer, 1982)

For all $n \geq 2$, there exists $\gamma_n < 1$ such that, for all compact manifold of dimension n,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma_n.$$

Proof for planar domains

Pleijel case :

- Let u, associated with $\lambda_k(\Omega)$, have ν_k nodal domains $D_1, \dots, D_{\nu_k}$.
- Applying Faber-Krahn : $\lambda_k(\Omega) = \lambda_1(D_i) \geq \frac{\pi j_{0,1}^2}{|D_i|}$ pour $1 \leq i \leq \nu_k$.
- Summing : $\nu_k \pi j_{0,1}^2 \leq \lambda_k(\Omega)|\Omega|$.
- Weyl's law : $\lambda_k(\Omega) \sim \frac{4\pi k}{|\Omega|}$.
- Conclusion : $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2} \simeq 0.6917.$

Remarks on the case of manifolds

Differences with the isoperimetric inequality in the plane : the manifold may have a curvature (sphere) or may not be simply connected (torus). Without any condition on the area, the isoperimetric ratio can be arbitrarily large.

We have two inequalities valid for domains with small area.

- For all $\varepsilon > 0$, there exists $A(\varepsilon, M) > 0$ such that, if $|D| \leq A(\varepsilon, M)$, $|\partial D||D|^{-\frac{n-1}{n}} \geq (1-\varepsilon)|\partial B||B|^{-\frac{n-1}{n}}$.
- If $|D| \leq A(\varepsilon, M)$, $\lambda_1(D)|D|^{\frac{2}{n}} \geq (1-\varepsilon)^2 \lambda_1(B)|B|^{\frac{2}{n}}$.

The Faber-Krahn type inequality is deduced from the asymptotic isoperimetric inequality by symmetrization, as in the euclidean case.

Proof in the case of manifolds

- We chose $\varepsilon > 0$ et $A = A(\varepsilon, M)$ as above.
- Let u be an eigenfunction associated with $\lambda_k(M)$ with ν_k nodal domains.
- There are at most $\ell = \lfloor |M|/A \rfloor$ nodal domains with volume greater than A.
- Let $(D_i)_{1 \leq i \leq N}$ be the nodal domains of volume less or equal than A.
- $\lambda_k(M) = \lambda_1(D_i) \geq (1-\varepsilon)^2 \lambda_1(B) |B|^{\frac{2}{n}} |D_i|^{-\frac{2}{n}}$ for $1 \leq i \leq N$.
- $\frac{\nu_k}{k} \leq \frac{|M| \lambda_k(M)^{\frac{n}{2}}}{k} (1-\varepsilon)^{-n} \lambda_1(B)^{-\frac{n}{2}} |B|^{-1} + \frac{\ell}{k}$.

$$\gamma_n = \frac{(2\pi)^n}{\beta_n j_{(n-1)/2,1}^n} = \frac{2^{n-2} n^2 \Gamma(\frac{n}{2})^2}{j_{(n-2)/2,1}^2} < 0.7.$$

Extensions and examples

Pleijel's result holds for eigenfunction of the Neumann Laplacian on domains in $\mathbb{R}^2$ having a piecewise analytic boundary (Polterovich, 2009).

In some specific examples, all Courant-sharp eigenvalues can be found :

- the square, Dirichlet case (Pleijel 1956, Bérard–Helffer 2014) ;
- the sphere (Leydold, 1996) ;
- the disk, Dirichlet case (Helffer–Hoffmann-Ostenhof–Terracini, 2009) ;
- the square, Neumann case (Helffer–Persson-Sundqvist, 2014) ;
- the square torus (L., 2014) ;
- the equilateral torus, the equilateral, hemi-equilateral and right angled isosceles triangles (Berard–Helffer, 2015)
- the disk, Neumann case (Helffer–Persson-Sundqvist, 2015) ;
- the cube, Dirichlet case (Helffer–Kiwan, 2015).

The problem can also be studied for the harmonic oscillator (Leydold, 1996, Bérard–Helffer, 2014)

Application : minimal partitions

- k-partition : family of k open, connected and disjoint subsets in M, $\mathcal{D} = \{D_1, \dots, D_k\}$.
- Energy : $\Lambda_k(\mathcal{D}) = \max_{1 \leq i \leq k} \lambda_1(D_i)$.
- $\mathfrak{L}_k(M) := \inf_{\mathcal{D}} \Lambda_k(\mathcal{D})$.
- Minimal k-partition $\mathcal{D}^*$: $\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(M)$.
- Nodal partition : familly of the nodal domains of an eigenfunction of $-\Delta_M$.
- A nodal partition associated with (λ, u) is minimal if, and only if, $\nu(u) = \kappa(\lambda)$ (Helffer–Hoffmann-Ostenhof–Terracini, 2009).

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Statement of the result

We denote by $\mathbb{T}^2$ the flat square torus of dimension 2

$$\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2.$$

Theorem

The only Courant-sharp eigenvalues of $-\Delta_{\mathbb{T}^2}$ are $\lambda_k(\mathbb{T}^2)$ for $k \in \{1, 2, 3, 4, 5\}$.

Corollary

The minimal k-partitions of $\mathbb{T}^2$ *are nodal only for* $k \in \{1, 2\}$.

Corollary

For $k \geq 3$, *we have* $\nu_k \leq k - 1$.

Isoperimetric inequality on tori

Theorem (Howards–Hutchings–Morgan, 1999)

Let $\mathbb{T}$ be a flat torus of dimension 2 whose closed geodesics have minimal length a. Let $0 < A < |\mathbb{T}|$. The region of $\mathbb{T}$ having area A and minimal perimeter is

- a (round) disk if $0 < A \leq \frac{a^2}{\pi}$;
- a strip bounded by geodesics if $\frac{a^2}{\pi} \leq A \leq |\mathbb{T}| - \frac{a^2}{\pi}$;
- the complement of a disk if $|\mathbb{T}| - \frac{a^2}{\pi} \leq A < |\mathbb{T}|$.

Proposition

Let $D \subset \mathbb{T}^2$ such that $|D| \leq \frac{1}{\pi}$. Then $\lambda_1(D)|D| \geq \pi j_{0,1}^2$.

Proof : we apply Schwartz symmetrization to the level sets $D_t = \{x\,;\, u(x) > t\}$, with u a positive eigenfunction associated with $\lambda_1(D)$.

Eigenvalues

Eigenvalues of $-\Delta_{\mathbb{T}^2}$:

$$\lambda_{m,n} = 4\pi^2(m^2 + n^2)$$

Eigenfunctions :

$$u^{cc}_{m,n}(x,y) = \cos(2m\pi x)\cos(2n\pi y);$$
$$u^{cs}_{m,n}(x,y) = \cos(2m\pi x)\sin(2n\pi y);$$
$$u^{sc}_{m,n}(x,y) = \sin(2m\pi x)\cos(2n\pi y);$$
$$u^{ss}_{m,n}(x,y) = \sin(2m\pi x)\sin(2n\pi y).$$

We associate to $\lambda_{m,n}$ a space of eigenfunctions $E_{m,n}$, of dimension 1, 2 or 4. We have

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n)\in\mathbb{N}^2} E_{m,n}}.$$

Weyl's law

- $N(\lambda) = \sharp\{k \,:\, \lambda_k(\mathbb{T}^2) \leq \lambda\}$ (counting function) ;
- $\mathcal{R}_\lambda = \left\{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\, y \geq 0 \text{ et } x^2 + y^2 \leq \frac{\lambda}{4\pi^2}\right\}$;
- $n(\lambda) = \sharp\left(\mathbb{N}^2 \cap \mathcal{R}_\lambda\right)$;
- $N(\lambda) = 4n(\lambda) - 4\left\lfloor \frac{\sqrt{\lambda}}{2\pi} \right\rfloor - 3$.

We obtain a lower bound on the counting function :

$$N(\lambda) \geq \frac{\lambda}{4\pi} - \frac{2\sqrt{\lambda}}{\pi} - 3.$$

We deduce :

$$\lambda_k(\mathbb{T}^2) \leq \left(4 + 2\sqrt{4 + \pi(k+3)}\right)^2.$$

Proof of the theorem

Notation : $j := j_{0,1}$

Remark : if λ is a Courant-sharp eigenvalue of $-\Delta_{\mathbb{T}^2}$ with $\kappa(\lambda) \geq 4$, then $\lambda \geq \pi j^2 \kappa(\lambda)$.

Direct computation : if

$$k > \frac{\left(4j + 2\sqrt{4j^2 + 3\pi(j^2 - 4)}\right)^2}{\pi(j^2 - 4)^2} \simeq 49.5973,$$

then

$$\left(4 + 2\sqrt{4 + \pi(k + 3)}\right)^2 < \pi j^2 k.$$

Conclusion : if $\kappa(\lambda) \geq 50$, the eigenvalue λ is not Courant-sharp.

$\frac{\lambda}{4\pi^2}$	indices	multiplicity	κ
0	$(0,0)$	1	1
1	$(1,0)$, $(0,1)$	4	2
2	$(1,1)$	4	6
4	$(2,0)$, $(0,2)$	4	10
5	$(2,1)$, $(1,2)$	8	14
8	$(2,2)$	4	22
9	$(3,0)$, $(0,3)$	4	26
10	$(3,1)$, $(1,3)$	8	30
13	$(3,2)$, $(2,3)$	8	38
16	$(4,0)$, $(0,4)$	4	46
17	$(4,1)$, $(1,4)$	8	50

TABLE : The 57 first eigenvalues

We have

$$\frac{\lambda_k(\mathbb{T}^2)}{4k\pi^2} < \frac{j^2}{4\pi} \simeq 0.4602.$$

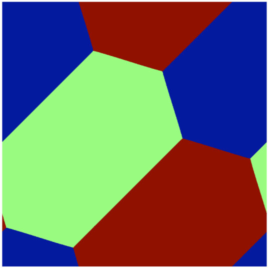
for $k \in \{6, 10, 14, 22, 26, 30, 38, 46\}$.

k	6	10	14	22	26	30	38	46
$\frac{\lambda_k(\mathbb{T}^2)}{4k\pi^2}$	0.3333	0.4000	0.3571	0.3636	0.3462	0.3333	0.3421	0.3478

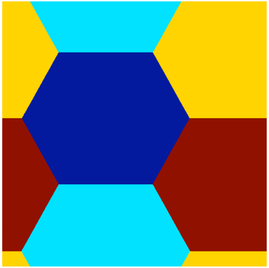
TABLE : Table of ratios

The only Courant-sharp eigenvalues are $\lambda_1(\mathbb{T}^2) = 0$ and $\lambda_2(\mathbb{T}^2) = \lambda_3(\mathbb{T}^2) = \lambda_2(\mathbb{T}^2) = \lambda_2(\mathbb{T}^2) = 4\pi^2$.

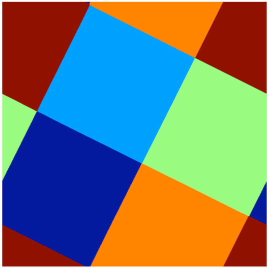
Numerical results



(a) $k = 3$



(b) $k = 4$



(c) $k = 5$

FIGURE : Minimal k-partitions of $\mathbb{T}^2$ for $k \in \{3, 4, 5\}$

Plan

$$\mathbb{T}_b^2 := (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$$

Eigenvalues of $-\Delta_{\mathbb{T}_b^2}$:

$$\lambda_{m,n} = 4\pi^2 \left(m^2 + \frac{n^2}{b^2} \right) ,$$

Eigenfunctions :

$$u^{cc}_{m,n}(x,y) = \cos(2\pi m x) \cos\left(\frac{2\pi n y}{b} \right) ;$$

$$u^{cs}_{m,n}(x,y) = \cos(2m\pi x) \sin\left(\frac{2n\pi y}{b} \right) ;$$

$$u^{sc}_{m,n}(x,y) = \sin(2m\pi x) \cos\left(\frac{2n\pi y}{b} \right) ;$$

$$u^{ss}_{m,n}(x,y) = \sin(2m\pi x) \sin\left(\frac{2n\pi y}{b} \right) .$$

Parity of the nodal count

T. Hoffmann-Ostenhof. Geometric aspects of spectral theory (July 1st – July 7th, 2012), Problem Section (xv).
Oberwolfach Rep., 9(3) :2013–2076, 2012.T. Hoffmann-Ostenhof.

Problem

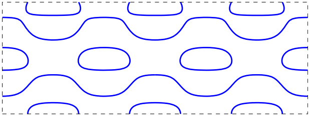
Is there a flat torus so that for some eigenfunction in the eigenspace $U(\lambda_{m,n})$ for some $\lambda_{m,n}$ there is an eigenfunction u with and odd number ≥ 3 of nodal domains ?

Irrational case

Proposition

If b^2 is irrational, any non-constant eigenfunction u of $-\Delta_{\mathbb{T}_b^2}$ has an even number of nodal domains.

Nodal lines of $u_{3,2}^{cc} + \frac{1}{2} u_{3,2}^{ss}$ for $b = e^{-1}$:



Proof :

- If $m > 0$, set $\mathbf{v} := (1/2m, 0)$.
- For each basis function $u \in \{u_{m,n}^{cc}, u_{m,n}^{cs}, u_{m,n}^{sc}, u_{m,n}^{ss}\}$, we have $u(\mathbf{x}+\mathbf{v}) = -u(\mathbf{x})$.
  Example : $\cos(2\pi mx + \pi)\cos\left(\frac{2\pi ny}{b}\right) = -\cos(2\pi mx)\cos\left(\frac{2\pi ny}{b}\right)$
- We have a bijection between positive and negative nodal domains.
- If $m = 0$, $n > 0$, and we do the same with $\mathbf{v} := (0, b/2n)$.

The case $b = 1$

Proposition

If $b = 1$, any non-constant eigenfunction u of $-\Delta_{\mathbb{T}^2_b}$ has an even number of nodal domains.

In that case we can have higher multiplicities with pairs $(m, n) \neq (m', n')$ such that $m^2 + n^2 = m'^2 + n'^2$.

Lemma (Hoffmann-Ostenhof, 2015)

For $(m, n) \neq (0, 0)$, we write $\lambda = m^2 + n^2$. If $\lambda = 2^{2p}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $m = 2^p m_0$ and $n = 2^p n_0$, where exactly one of the integers m_0 and n_0 is odd. If $\lambda = 2^{2p+1}(2q + 1)$ with $(p, q) \in \mathbb{N}^2$, then $n = 2^p m_0$ and $n = 2^p n_0$, where both integers m_0 and n_0 are odd.

- In the first case, we set $\mathbf{v} := (1/2^{p+1}, 1/2^{p+1})$.
- In the second case, we set $\mathbf{v} := (1/2^{p+1}, 0)$ or $\mathbf{v} := (0, 1/2^{p+1})$.

Corollary

For $k \geq 4$, we have $\nu_k \leq k - 2$.

Counter-example for $b = \frac{1}{\sqrt{3}}$

Proposition

If $b = \frac{1}{\sqrt{3}}$, there exists an eigenfunction of $-\Delta_{\mathbb{T}_b^2}$ with three nodal domains.

Idea : consider $v_\varepsilon = u_{1,1}^{cc} + \varepsilon u_{2,0}^{cc}$.

(a) $\mathcal{N}(u_{1,1}^{cc})$ (b) $\mathcal{N}(u_{2,0}^{cc})$

FIGURE : Nodal sets of basis functions

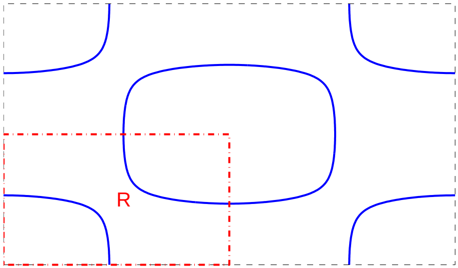


FIGURE : Nodal set of $v_\varepsilon = u^{cc}_{1,1} + \varepsilon u^{cc}_{2,0}$ with $\varepsilon = 0.1$

Proof

$$R := \left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2\sqrt{3}}\right[.$$

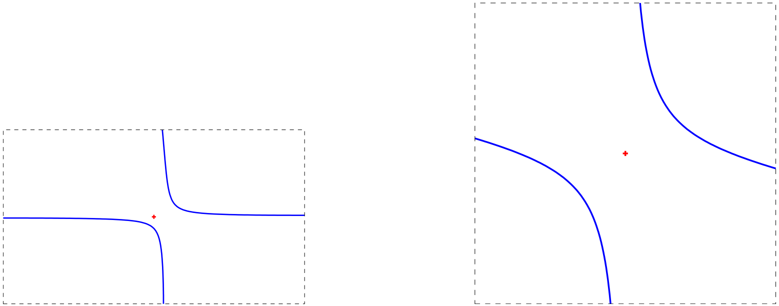
Change of coordinates :

$$\begin{cases} s = -\cos(2\pi x) ; \\ t = -\cos\left(\frac{2\pi y}{b}\right) . \end{cases}$$

R is sent to $]-1,1[\times]-1,1[$. Nodal set of v_ε in the new coordinates :

$$st + \varepsilon(2s^2 - 1) = 0 ,$$

two branches of hyperbola.



(a) In (x, y)-coordinates

(b) In (u, v)-coordinates

FIGURE : Nodal set in R

Open problem

- The arithmetical lemma still holds for the equation $\lambda = \alpha m^2 + \beta n^2$, with $\alpha + \beta = 2 \bmod 4$. Therefore, if $b = \sqrt{\frac{\alpha}{\beta}}$ there can only be an even number of nodal domains.
- Assuming that

$$m^2 + \frac{n^2}{b^2} = k^2 m^2,$$

that is to say

$$b = \frac{n}{m\sqrt{k^2 - 1}},$$

$$v_\varepsilon = u^{cc}_{m,n} + \varepsilon u^{c,c}_{km,0}$$

is an eigenfunction, with $2mn + 1$ nodal domains for a small ε.
- Problem : characterizing the rational numbers q, such that there exists an eigenfunction on the torus $\mathbb{T}^2_{\sqrt{q}}$ with an odd number of nodal domains.