Minimal partitions of flat tori

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Outline of the talk

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Definitions and notation

- Ω a two-dimensional domain in $\mathbb{R}^2$ or in a two-dimensional Riemannian manifold.
- k-partition of Ω : a family of k disjoint, open, and connected subsets of Ω :

$$\mathcal{D} = (D_i)_{1 \leq i \leq k}.$$

- If $D \subset \Omega$, $\lambda_1(D)$ first eigenvalue of the Dirichlet Laplacian in D.
- Energy of a k-partition : $\Lambda(\mathcal{D}) = \max_{1 \leq i \leq k} \lambda_1(D_i)$.
- Minimal energy of a k-partition of Ω : $\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D}} \Lambda(\mathcal{D})$.
- Minimizers exist and are regular (valid for more general energies).
- In this example, minimal partitions are equispectral :

$$\lambda_1(D_1) = \cdots = \lambda_1(D_k).$$

Connection with nodal partitions

Let u be an eigenfunction of $-\Delta_\Omega$:

- $\mathcal{N}(u) = \overline{u^{-1}(0)}$ is called the nodal set of u ;
- the connected components of $\Omega \setminus \mathcal{N}(u)$ are called the nodal domains of u ;
- $\mu(u)$ denotes the number of nodal domains ;
- the family of nodal domains is called the nodal partition associated with u.

Additional notation :

$$L_k(\Omega) = \inf\{\lambda : \text{ there exists } u \text{ with } -\Delta_\Omega\, u = \lambda\, u \text{ and } \mu(u) = k\}.$$

Theorem (Helffer–Hoffmann-Ostenhof–Terracini, 2009)

For all k,

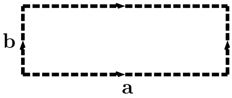
$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega).$$

If $\lambda_k(\Omega) = \mathfrak{L}_k(\Omega)$ or $\mathfrak{L}_k(\Omega) = L_k(\Omega)$, then

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega).$$

Examples : flat tori

For $0 < b \leq a$, we write $T(a,b) := (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$.



For a given integer k, we consider minimal k-partitions of $T(1,b)$, and in particular how they depend on $b \in (0,1]$. We are interested in the following questions.

- Can we describe minimal partitions ? In particular, can we find explicit examples of non-nodal minimal partitions ?
- Conversely, when are minimal partitions nodal ? This amounts to describing Courant-sharp situations.
- Can we find $\mathfrak{L}_k(T(1,b))$, or at least upper and lower bounds ?

Eigenvalues and eigenfunctions

Eigenvalues of $-\Delta_{T(a,b)}$: $\lambda_{m,n}(a,b) = 4\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$.
Associated eigenfunctions :

$$u^{cc}_{m,n}(x,y) = \cos\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{b}\right);$$

$$u^{cs}_{m,n}(x,y) = \cos\left(\frac{2m\pi x}{a}\right)\sin\left(\frac{2n\pi y}{b}\right);$$

$$u^{sc}_{m,n}(x,y) = \sin\left(\frac{2m\pi x}{a}\right)\cos\left(\frac{2n\pi y}{b}\right);$$

$$u^{ss}_{m,n}(x,y) = \sin\left(\frac{2m\pi x}{a}\right)\sin\left(\frac{2n\pi y}{b}\right).$$

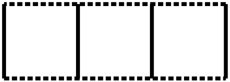
$E_{m,n}(a,b)$ is a space of eigenfunctions associated with $\lambda_{m,n}(a,b)$, with dimension 1, 2, or 4. $L^2(T(a,b)) = \overline{\bigoplus_{(m,n)\in\mathbb{N}^2} E_{m,n}(a,b)}$.

Outline of the talk

Definitions

- $\mathcal{D}_k(1, b)$ is the k-partition of $T(1, b)$ with domains

$$D_i = \left(\frac{i-1}{k}, \frac{i}{k}\right) \times (0, b) \;, \qquad \text{for } i = 1, \dots, k \,.$$



- $\Lambda(\mathcal{D}_k(1, b)) = k^2\pi^2$.
- Transition value : $b_k = \sup\{b > 0 \; ; \; \forall b' \in (0, b) \,, \mathfrak{L}_k(T(1, b')) = k^2\pi^2\} \,.$
- Alternative definition : $b_k = \sup\{b > 0 \; ; \; \mathfrak{L}_k(T(1, b)) = k^2\pi^2\}$. Equality comes from the fact that $b \mapsto \mathfrak{L}_k(T(1, b))$ is non-increasing.

Case of an even k

We write $k = 2\ell$. Then $\mathcal{D}_k(1, b)$ is the nodal partition given by the eigenfunction

$$u^{sc}_{\ell,0}(x, y) = \sin(2\ell\pi x) \ ,$$

associated with the eigenvalue $\lambda_{\ell,0}(1, b) = 4\ell^2\pi^2 = k^2\pi^2$. According to the general theorem, $\mathcal{D}_k(1, b)$ is minimal if, and only if, $\lambda_{\ell,0}(1, b) = \lambda_k(T(1, b))$. This occurs when

$$4\ell^2\pi^2 = \lambda_{\ell,0}(1, b) \leq \lambda_{1,b}(1, b) = \frac{4\pi^2}{b^2} \, ,$$

i.e. when

$$b \leq \frac{1}{\ell} = \frac{2}{k}.$$

We have obtained $b_k = \frac{2}{k}$ for an even k.

Case of an odd k

Theorem (Helffer–Hoffmann-Ostenhof, 2014)

If k is odd, $b_k \geq \frac{1}{k}$.

- If k is odd, $\mathcal{D}_k(1, b)$ is not a nodal partition (coloring argument). This gives an explicit example of a non-nodal minimal partition.
- The proof uses the following ideas :
  - covering arguments ;
  - Steiner symmetrization ;
  - study of the topological type of the partition ;
  - going back to the even case.
- Not optimal.

Conjecture

$$b_k = \frac{2}{\sqrt{k^2 - 1}}.$$

Proof I

Lemma

If $D \subset T(1,b)$ is homeomorphic to a disk, $\lambda_1(D) \geq \frac{\pi^2}{b^2}$.

Proof :

- Covering

$$\begin{array}{cccc} \Pi_\infty : & \mathbb{R}^2 & \to & T(1,b) \\ & (x,y) & \mapsto & (x \bmod 1, y \bmod b). \end{array}$$

- D_0 one of the connected components of $\Pi_\infty^{-1}(D)$.
- D_0^* Steiner symmetrization with respect to $\{y=0\}$, contained in $S_b = \mathbb{R}\times]-b/2, b/2[$.
- Domain monotonicity of λ_1 :

$$\frac{\pi^2}{b^2} \leq \lambda_1(D_0^*) \leq \lambda_1(D_0) = \lambda_1(D).$$

Consequence : if $b < 1/k$, a minimal partition of $T(1,b)$ has no disk.

Proof II

- Covering

$$\begin{array}{rccc} \Pi : & T(2,2b) & \to & T(1,b) \\ & (x,y) & \mapsto & (x \bmod 1, y \bmod b). \end{array}$$

- A partition $\mathcal{D}$ of $T(1,b)$ gives a partition $\Pi^{-1}(\mathcal{D})$ of $T(2,2b)$.
- Domains : connected components of the pullbacks of domains.
- Same energy.

Lemma

If $\mathcal{D}$ is a regular k-partition of $T(1,b)$ without any disk, $\Pi^{-1}(\mathcal{D})$ is a $2k$-partition of $T(2,2b)$.

Conclusion : Let $b < 1/k$ and $\mathcal{D}$ a minimal k- partition of $T(1,b)$. $\mathcal{D}$ has no disk, so $\Pi^{-1}(\mathcal{D})$ is a $2k$-partition of $T(2,2b)$. From the even case : since $b < \frac{1}{k} = \frac{2}{2k}$, $\Lambda(\mathcal{D}) = \Lambda(\Pi^{-1}(\mathcal{D})) \geq \Lambda(\mathcal{D}_{2k}(2,2b)) = k^2\pi^2$.

Antisymmetric eigenfunctions

- We define the symmetry $\sigma : T(2,2b) \to T(2,2b)$ by

$$\sigma(x,y) = (x+1 \bmod 2, y).$$

- u symmetric if $u \circ \sigma = u$, antisymmetric if $u \circ \sigma = -u$.
- $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) space of symmetric (resp. antisymmetric) functions.
- Orthogonal decomposition :

$$L^2(T(2,2b)) = \mathcal{S}^+ \oplus \mathcal{S}^-.$$

- Since $-\Delta(u \circ \sigma) = (-\Delta u) \circ \sigma$, $(\lambda_k(T(2,2b))) = (\lambda_k^+) \cup (\lambda_k^-)$, with λ_k^+ spectrum of $(-\Delta)_{|\mathcal{S}^+}$ and λ_k^- spectrum of $(-\Delta)_{|\mathcal{S}^-}$.
- If u is antisymmetric, the nodal domains can be divided into pairs (D^+, D^-) with $u > 0$ on D^+, $u < 0$ on D^-, $\sigma(D^+) = D^-$ and $\Pi(D^+) = \Pi(D^-)$.

Antisymmetric partitions

Based on ideas used by Helffer–Hoffmann-Ostenhof–Terracini (2009) in the case of the sphere.

Definition

A regular partition $\mathcal{D} = (D_i)_{1 \leq i \leq N}$ of $T(2, 2b)$ is antisymmetric when $\sigma(D_i) = D_j$ with $j \neq i$ for all $1 \leq i \leq N$.

In that case, $N = 2k$ and the domains can be divided into k pairs (D_i^+, D_i^-) with $\sigma(D_i^+) = D_i^-$.

Proposition

Let $\mathcal{D}$ be a regular antisymmetric $2k$-partition. We have $\lambda_k^- \leq \Lambda(\mathcal{D})$.

This is a refined version of Courant theorem. It is proved by applying the variational characterization of eigenvalues in the space of antisymmetric function.

Auxiliary optimization problem

We look for

$$J(b, A) = \inf_{\Omega \subset S_b, |\Omega| \leq A} \lambda_1(\Omega),$$

with

$$S_b = \mathbb{R} \times]-b/2, b/2[.$$

- $J(b, A)$ is non-increasing with respect to b and A.
- $J(b, A) \geq \frac{\pi^2}{b^2}$ (domain monotonicity).
- There exists a quasi-open set Ω^* such that $\lambda_1(\Omega^*) = J(b, A)$, and $|\Omega^*| = A$ (from the concentration-compactness for shapes result of Bucur, 2000).

Application

We look for

$$b_k^{FK} = \sup\left\{ b \in (0,1] \;:\; J(b,b) > k^2\pi^2 \right\}.$$

Lemma

$$\frac{1}{k} < b_k^{FK} < \frac{1}{\sqrt{k^2-1}}$$

Proof :

- $b \mapsto J(b,b)$ continuous (elementary analysis).
- $J(1/k, 1/k) > k^2\pi^2$ (from the existence result).
- We get $b_k^{FK} > 1/k$.
- $J(b,b) < \pi^2\left(1+\frac{1}{b^2}\right)$ (a rectangle of size $1 \times b$ is not minimal).
- We get $b_k^{FK} < 1/\sqrt{k^2-1}$.

Conclusion

Proposition

If k is odd and $b < b_k^{FK}$, then $\mathcal{D}_k(1, b)$ is a minimal partition of $T(1, b)$.

Proof : Let $\mathcal{D}$ be a minimal partition of $T(1, b)$.

- $\mathcal{D}$ has no disk.
- $\Pi^{-1}(\mathcal{D})$ is a $2k$-partition of $T(2, 2b)$.
- $\Pi^{-1}(\mathcal{D})$ is equispectral with energy $\Lambda(\mathcal{D})$ and is antisymmetric, thus

$$\lambda_k^- \leq \Lambda(\mathcal{D}).$$

The antisymmetric eigenvalues of $-\Delta_{T(2,2b)}$ are the $\lambda_{2p+1,q}(2, 2b)$'s. Let us write $k = 2\ell + 1$. Since $b < \frac{1}{\sqrt{k^2-1}}$, a direct computation shows that $\lambda_k^- = k^2\pi^2$.

Outline of the talk

Statement of the problem

If λ eigenvalue of $-\Delta_\Omega$, $\nu(\lambda) := \min\{k \geq 1\,;\, \lambda = \lambda_k(\Omega)\}$.

Courant Theorem

If u is an eigenfunction $-\Delta_\Omega$ associated with the eigenvalue λ, $\mu(u) \leq \nu(\lambda)$.

Definition

An eigenfunction u, associated with the eigenvalue λ is called Courant-sharp if $\mu(u) = \nu(\lambda)$.

Theorem (Pleijel, 1956)

If Ω is an bounded open set in $\mathbb{R}^2$, only a finite number of eigenvalues of $-\Delta_\Omega$ have Courant-sharp eigenfunctions.

Improvements : the analogous result holds for a n-dimensional compact Riemannian manifold M, with or without boundary (Bérard–Meyer, 1982) ; in particular (solvable) cases, we know exactly which eigenvalues have Courant-sharp eigenfunctions (Bérard–Helffer–Persson Sundqvist).

The irrational case : twisting trick

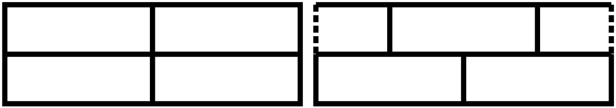
Theorem (Helffer–Hoffmann-Ostenhof, 2014)

Suppose that b^2 is irrational. If $m \geq 1$ and $n \geq 1$, the eigenvalue $\lambda_{m,n}(1,b)$ has no Courant-sharp eigenfunction.

- The number of nodal domains is maximal (equal to $m \times n$) for products :

$$(x,y) \mapsto \lambda \cos(2\pi x + \theta) \cos\left(\frac{2\pi y}{b} + \varphi\right) .$$

- The nodal partition associated with the eigenfunction above is not minimal (twisting trick).



Isotropic case

Theorem

Only the eigenvalues of $-\Delta_{T(1,1)}$ are $\lambda_k(T(1,1))$ for $k \in \{1,2,3,4,5\}$ have Courant-sharp eigenfunction.

Corollary

For $k \geq 3$, the minimal k-partitions of $T(1,1)$ are not nodal, and thus

$$\lambda_k(T(1,1)) < \mathfrak{L}_k(T(1,1)) < L_k(T(1,1)).$$

Isoperimetric inequalities

Theorem (Howards–Hutchings–Morgan, 1999)

Let $\mathbb{T}$ be a two-dimensional flat torus whose closed geodesics have minimal length a. Let $0 < A \leq \frac{a^2}{\pi}$. The a (circular) disks have minimal perimeter.

Proposition

Let D be an open set in $T(1,1)$ such that $|D| \leq \frac{1}{\pi}$. Then

$$\lambda_1(D) \geq \frac{\pi j_{0,1}^2}{|D|}.$$

The proof uses Schwarz symmetrization of the level sets into disks.

Explicit Weyl's law

- $N(\lambda) = \sharp\{k \,:\, \lambda_k(T(1,1)) \leq \lambda\}$ (counting function) ;
- $\mathcal{R}_\lambda = \left\{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\, y \geq 0 \text{ et } x^2 + y^2 \leq \frac{\lambda}{4\pi^2}\right\}$;
- $n(\lambda) = \sharp\left(\mathbb{N}^2 \cap \mathcal{R}_\lambda\right)$;
- $N(\lambda) = 4n(\lambda) - 4\left\lfloor \frac{\sqrt{\lambda}}{2\pi} \right\rfloor - 3.$

We get a lower bound for the counting function :

$$N(\lambda) \geq \frac{\lambda}{4\pi} - \frac{2\sqrt{\lambda}}{\pi} - 3.$$

We deduce :

$$\lambda_k(T(1,1)) \leq \left(4 + 2\sqrt{4 + \pi(k+3)}\right)^2.$$

Sketch of the proof

Remark : if λ is an eigenvalue of $-\Delta_{T(1,1)}$ with $\nu(\lambda) \geq 4$, which has a Courant-sharp eigenfunction, then $\lambda \geq \pi j_{0,1}^2 \nu(\lambda)$.

A direct computation using the upper bound for $\lambda_k(T(1,1))$ implies : if $\nu(\lambda) \geq 50$, λ has no Courant-sharp eigenfunctions.
Indeed, if u is a Courant-sharp eigenfunction, one of its domain D satisfies $|D| \leq 1/\nu(\lambda)$. Then

$$\lambda = \lambda_1(D) \geq \frac{\pi j_{0,1}}{|D|} \geq \pi j_{0,1} \nu(\lambda).$$

We can compute the ratio $\frac{\lambda}{\nu(\lambda)}$ for the finite number of remaining eigenvalues, and check that it is strictly smaller than $\pi j_{0,1}^2$. Only the eigenvalues

$$\lambda_1(T(1,1)) = 0$$

and

$$\lambda_2(T(1,1)) = \lambda_3(T(1,1)) = \lambda_4(T(1,1)) = \lambda_5(T(1,1)) = 4\pi^2$$

have Courant-sharp eigenfunctions.

Outline of the talk

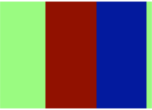
Objectives of the numerical study

- We look for minimal k-partitions of $T(1, b)$, according to the parameter b, with $k \in \{3, 4, 5\}$.
- We know that $b_4 = 1/2$. We conjecture

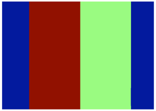
$$\begin{aligned} b_3 &= \frac{1}{\sqrt{2}} &\simeq 0.7071\,; \\ b_5 &= \frac{1}{\sqrt{6}} &\simeq 0.4082\,. \end{aligned}$$

- We want to study minimal partitions for b close to these transition values, to understand the transition.
- The method is based on work by Bourdin–Bucur–Oudet (2009).
- Work in collaboration with Virginie Bonnaillie-Noël.

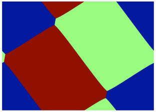
3-partitions



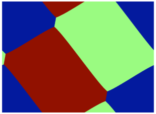
(c) $b = 0.70$



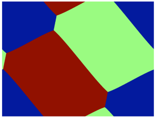
(d) $b = 0.71$



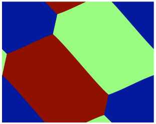
(e) $b = 0.72$



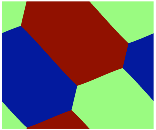
(f) $b = 0.73$



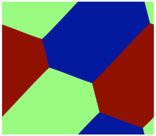
(g) $b = 0.76$



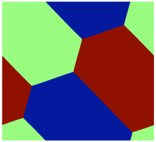
(h) $b = 0.80$



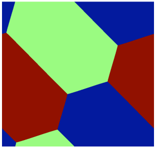
(i) $b = 0.84$



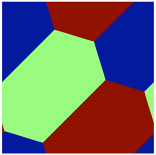
(j) $b = 0.88$



(k) $b = .92$

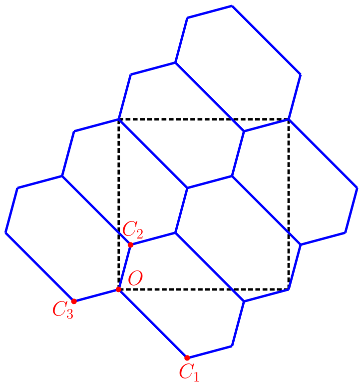


(l) $b = 0.96$



(m) $b = 1$

Hexagonal tilings



If we compute λ_1 for the tilling domain by a finite element method, we get a good upper bound for $\mathfrak{L}_3(T(1,b))$.

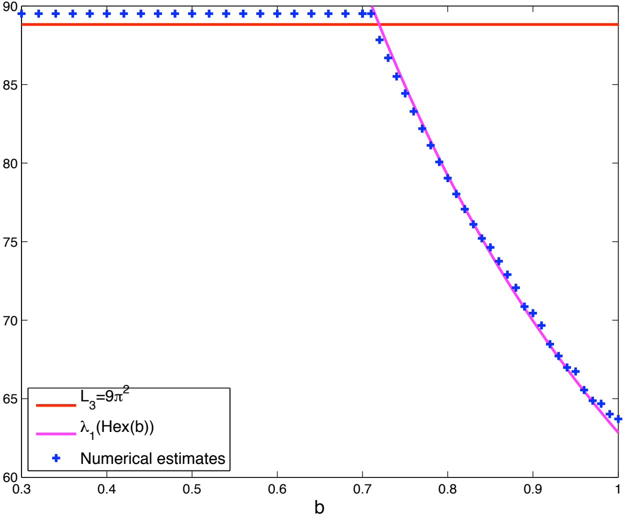


FIGURE : Upper bound of $\mathfrak{L}_3(T(1,b))$ as a function of b

Nodal partitions of $T(1, 1/2)$

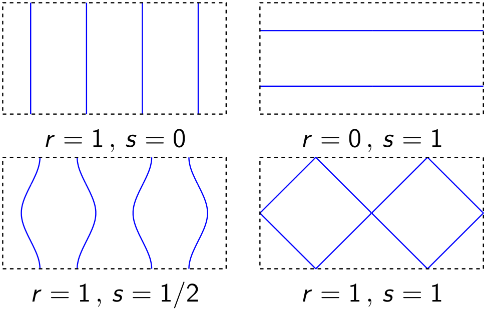


FIGURE : Nodal lines of $u(x, y) = r\cos(4\pi x) + s\cos(4\pi y)$

The nodal 4-partitions are minimal, with energy $\lambda_4(T(1, 1/2)) = 16\pi^2$.
Conjectured deformation mechanism : splitting of the singular point of order 4 into pairs of singular points of order 3.

4-partitions

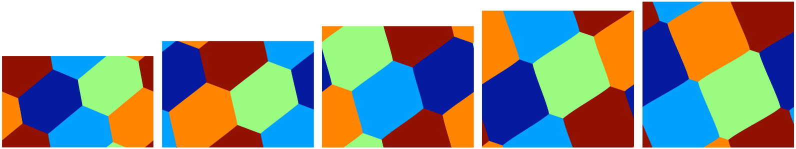
(a) $b = 0.49$ (b) $b = 0.50$ (c) $b = 0.51$ (d) $b = 0.52$ (e) $b = 0.55$

(f) $b = 0.60$ (g) $b = 0.7$ (h) $b = 0.8$ (i) $b = 0.9$ (j) $b = 1$

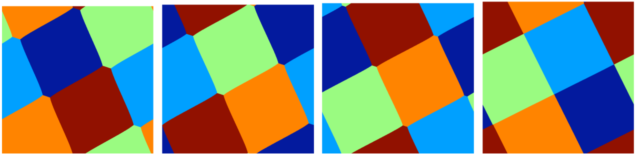
Remark : the equal angle property should imply that the boundary of the partition is curved in the neighborhood of the critical point.

5-partitions

(k) $b = 0.40$ (l) $b = 0.41$ (m) $b = 0.42$ (n) $b = 0.43$ (o) $b = 0.5$



(p) $b = 0.6$ (q) $b = 0.7$ (r) $b = 0.8$ (s) $b = 0.9$ (t) $b = 0.96$



(u) $b = 0.97$ (v) $b = 0.98$ (w) $b = 0.99$ (x) $b = 1$

Candidate 5-partition of $T(1,1)$

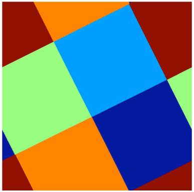


FIGURE : $\Lambda = 99.05$

The partition of $T(1,1)$ into five squares has energy $\Lambda_5^{sq} = 10\pi^2 \simeq 98.6960$.
It is not bipartite , and therefore not nodal.
To make it nodal, two equivalent possibilities :

- lift it on the 4-fold covering $T(2,2)$ of $T(1,1)$ (20-partition) ;
- add a vector potential $\mathbf{A}$ (with rot $\mathbf{A} = 0$) that has circulation π on the two non-trivial loops.

Problems and conjectures

- Is $b_k = 2/\sqrt{k^2-1}$? New idea to show that we obtain a $2k$-partition on $T(2,b)$?
- Hexagonal 3-partition or square 5-partition of $T(1,1)$? Can we just show that there exists a vector potential on $T(1,1)$ (or a covering) that "nodalizes" a minimal partition ? The topology of $T(1,1)$ should play a part. General framework for surfaces ?
- Can we make the perturbation argument near the transition values rigorous ? We should have some continuity of minimal partitions with respect to the domain.
  - How do we measure the distance between two partitions ?
  - Problem caused by non-uniqueness.