Can one hear the shape of a drum?

After M. Kac, C. Gordon, D. Webb, and S. Wolpert

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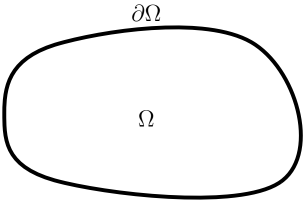
Motivation: the wave equation

$\Omega \subset \mathbb{R}^2$ open, bounded, and piecewise regular open set (think of a membrane fixed at its boundary). We call it a domain.
$z(t, x, y)$ vertical displacement of the membrane at the point $(x, y) \in \Omega$ for the time t.



z satisfies the wave equation :

$$\frac{\partial^2 z}{\partial t^2}(t, x, y) = c^2 (\Delta z)(t, x, y) \text{ for } (x, y) \in \Omega,$$

with the Laplacian Δ defined by:

$$(\Delta f)(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y),$$

and the Dirichlet boundary condition:

$$z(t, x, y) = 0 \text{ for } (x, y) \in \partial\Omega.$$

The constant c depends on the physical properties of the membrane. We chose the units so that $c = 1$.

Motivation: stationary solutions

Stationary solutions, or pure tones: each point of the membrane goes up and down in an harmonic motion. We therefore look for solutions of the form

$$z(t,x,y) = u(x,y)\cos(\omega(t-t_0)).$$

The function u must satisfy

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega; \qquad (1) \\ u &= 0 \text{ on } \partial\Omega. \qquad (2) \end{aligned}$$

with $\lambda = \omega^2$.
We say that λ is a Dirichlet eigenvalue if there exists a non-zero solution u of (1) and (2). We call u a eigenfunction associated with λ, and we call eigenspace associated with λ the vector space of all eigenfunctions.

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If we look at membranes that are free to move up and down at the boundary, we obtain the Neumann eigenvalues:

$$\begin{aligned} -\Delta u &= \mu u \text{ in } \Omega; & (3)\\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega. & (4) \end{aligned}$$

Example of a square membrane

We assume $\Omega = (0, L)^2$ with $L > 0$ (square membrane of side L).
For any pair (m, n) of positive integers,

$$u_{m,n}(x, y) := \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right)$$

is an eigenfunction, associated to the eigenvalue

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With some Fourier analysis, we can show that any eigenvalue is of the form $\lambda_{m,n}$ for some $(m, n) \in \mathbb{N}_1^2$, and that an eigenfunction for the Dirichlet problem associated with $\lambda_{m,n}$ is a linear combination of the functions $u_{p,q}$ satisfying $p^2 + q^2 = m^2 + n^2$. In particular, all the eigenspaces are finite dimensional.

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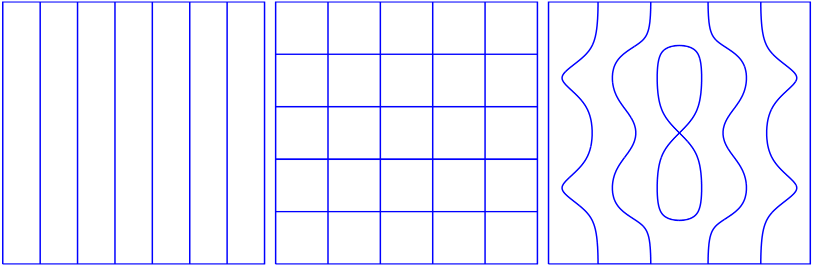
For the Neumann eigenvalue problem, we find

$$u_{m,n}(x, y) := \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right) \text{ and } \mu_{m,n} = \frac{\pi^2}{L^2}(m^2 + n^2)$$

with $(m, n) \in \mathbb{N}_0^2$. The same results are true.

Some examples for $50 = 7^2 + 1^2 = 5^2 + 5^2$



(a) $m = 7$ and $n = 1$ (b) $m = 5$ and $n = 5$ (c) Linear combination

Generalization

Theorem

For a general domain Ω, there exists countably many eigenvalues, and each of the associated eigenspace is finite dimensional.
If we write the eigenvalues

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \ldots,$$

repeating them according to the dimensions of the eigenspaces, we have $\lim_{k\to+\infty} \lambda_k(\Omega) = +\infty$.
We can find a sequence of associated eigenfunctions $u_1, u_2, \ldots, u_k, \ldots$ *such that every solution* z *of the wave equation can be written*

$$z(t,x,y) = \sum_{k=1}^{+\infty} A_k \cos(\sqrt{\lambda_k}t + \varphi_k) u_k(x,y).$$

We have the same result for the Neumann eigenvalues, with the sequence $(\mu_k(\Omega))_{k\geq 1}$. The sequence of all the eigenvalues for a domain is called its (Dirichlet or Neumann) spectrum.

Direct and converse spectral problems

We deduce from the preceding theorem that

One can see the sound of a drum.

In particular, if two domains Ω_1 and Ω_2 are isometric, that is to say if there exists an isometry $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ (a length-preserving transformation) such that $\tau(\Omega_1) = \Omega_2$, then Ω_1 and Ω_2 are (Dirichlet-)isospectral: $\lambda_k(\Omega_1) = \lambda_k(\Omega_2)$ for all $k \geq 1$.

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In its 1966 paper, Mark Kac asked the converse question: if we have two domains Ω_1 and Ω_2 that are Dirichlet-isospectral, are they isometric? This would mean that the spectrum of a domain determines its shape.

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In its 1966 paper, Mark Kac asked the converse question: if we have two domains Ω_1 and Ω_2 that are Dirichlet-isospectral, are they isometric? This would mean that the spectrum of a domain determines its shape. In other words:

Can one hear the shape of a drum?

Weyl's law

We define the counting functions associated with a domain:

$$\begin{aligned} N^D(\lambda,\Omega) &:= \sharp\{k : \lambda_k(\Omega) < \lambda\}; \\ N^N(\lambda,\Omega) &:= \sharp\{k : \mu_k(\Omega) < \lambda\}. \end{aligned}$$

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In particular,

$$N^D(\lambda, (0, L)^2) = \sharp\left\{(m, n) \in \mathbb{N}_1^2 : \frac{\pi^2}{L^2}(m^2 + n^2) < \lambda\right\};$$
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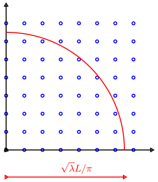
In that case we find, when $\lambda \to +\infty$,

$$N^D(\lambda,(0,L)^2) \sim N^N(\lambda,(0,L)^2) \sim \frac{L^2}{4\pi}\lambda = \frac{\text{Area}(\Omega)}{4\pi}\lambda.$$



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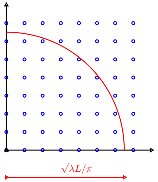

Theorem (Hermann Weyl, 1911)

For a general domain,

$$N^D(\lambda,\Omega) \sim N^N(\lambda,\Omega) \sim \frac{Area(\Omega)}{4\pi}\lambda \text{ when } \lambda \to +\infty.$$

Proof of Weyl's law : basic principles

We now prove Weyl's law for the Dirichlet eigenvalues.

Disjoint union of domains

If $\Omega = \Omega' \cup \Omega''$, the union being disjoint

$$\begin{aligned}(\lambda_k(\Omega))_{k\geq 1} &= (\lambda_k(\Omega'))_{k\geq 1} \cup (\lambda_k(\Omega''))_{k\geq 1} \\ (\mu_k(\Omega))_{k\geq 1} &= (\mu_k(\Omega'))_{k\geq 1} \cup (\mu_k(\Omega''))_{k\geq 1}\end{aligned}$$

(with repetition), and therefore

$$N^{D/N}(\lambda, \Omega) = N^{D/N}(\lambda, \Omega') + N^{D/N}(\lambda, \Omega'').$$

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$$N^{D/N}(\lambda, \Omega) = N^{D/N}(\lambda, \Omega') + N^{D/N}(\lambda, \Omega'').$$

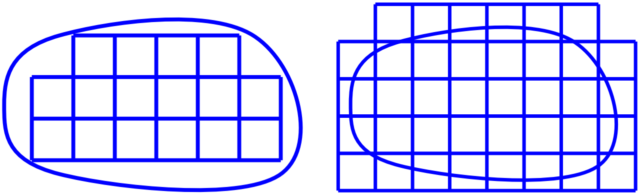
Monotonicity principle (Richard Courant)

If we add a Dirichlet condition, all the eigenvalues go up, and therefore the counting function goes down.
If we replace a Dirichlet condition by a Neumann condition, or if we add a Neumann condition, all the eigenvalues go down, and therefore the counting function goes up.

Proof of Weyl's law: inner and outer approximation

For a given $\varepsilon > 0$, we find $\ell > 0$, Ω_i and Ω_o such that

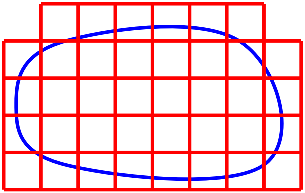
- Ω_i is the reunion of n_i squares of side ℓ and Ω_0 is the reunion of n_o squares of side ℓ;
- $\Omega_i \subset \Omega \subset \Omega_o$;
- $Area(\Omega_i) \geq (1-\varepsilon)Area(\Omega)$ and $Area(\Omega_o) \geq (1+\varepsilon)Area(\Omega)$.



$$N^D(\lambda,\Omega) \geq N^D(\lambda,\Omega_i) = n_i N(\lambda,(0,\ell)^2) \sim \frac{n_i \ell^2}{4\pi}\lambda \geq \frac{(1-\varepsilon)Area(\Omega)}{4\pi}\lambda$$

Proof of Weyl's law: Dirichlet-Neumann bracketing

For the outer approximation, we put a Neumann boundary condition on the boundary of the squares.



$$N^D(\lambda,\Omega) \leq N^N(\lambda,\Omega_o) = n_o N^N(\lambda,(0,\ell)^2) \sim \frac{n_o \ell^2}{4\pi}\lambda \leq \frac{(1+\varepsilon)|\Omega|}{4\pi}\lambda$$

We conclude that

$$\frac{(1-\varepsilon)|\Omega|}{4\pi} \leq \liminf_{\lambda\to+\infty} \frac{N^D(\lambda,\Omega)}{\lambda} \leq \limsup_{\lambda\to+\infty} \frac{N^D(\lambda,\Omega)}{\lambda} \leq \frac{(1+\varepsilon)|\Omega|}{4\pi}.$$

We make $\varepsilon \to 0$ and we obtain Weyl's law.

The Rayleigh-Faber-Krahn inequality I

Lord Rayleigh, 1877, in *The theory of sound*, volume I

Of all membranes of equal area that of circular form as the gravest pitch.

Circle	$2{\cdot}404 \,.\, \sqrt{\pi} = 4{\cdot}261.$
Square	$\sqrt{2} \,.\, \pi = 4{\cdot}443.$
Quadrant of a circle	$\frac{5{\cdot}135}{2} \,.\, \sqrt{\pi} = 4{\cdot}551.$
Sector of a circle 60°	$6{\cdot}379 \sqrt{\frac{\pi}{6}} = 4{\cdot}616.$
Rectangle 3 × 2	$\sqrt{\frac{13}{6}} \,.\, \pi = 4{\cdot}624.$
Equilateral triangle	$2\pi \,.\, \sqrt{\tan 30^\circ} = 4{\cdot}774.$
Semicircle	$3{\cdot}832 \sqrt{\frac{\pi}{2}} = 4{\cdot}803.$
Rectangle 2 × 1 / Right-angled isosceles triangle	$\pi \sqrt{\frac{5}{2}} = 4{\cdot}967.$
Rectangle 3 × 1	$\pi \sqrt{\frac{10}{3}} = 5{\cdot}736.$

The Rayleigh-Faber-Krahn inequality II

Theorem (Georg Faber 1923, Edgar Krahn 1925-1926)

If D_Ω is a disk with the same area as the domain Ω,

$$\lambda_1(D_\Omega) \leq \lambda_1(\Omega),$$

with equality if, and only if, Ω is isometric to D_Ω.

This inequality can be deduced from the isoperimetric inequality.
Consequence:

One can hear the shape of a circular drum.

More precise formulation:

Corollary (Spectral rigidity of the disk)

If a domain Ω as the same eigenvalues as a disk D, then Ω is isometric to D.

One cannot hear the shape of a drum

Conjecture (Mark Kac, 1966)

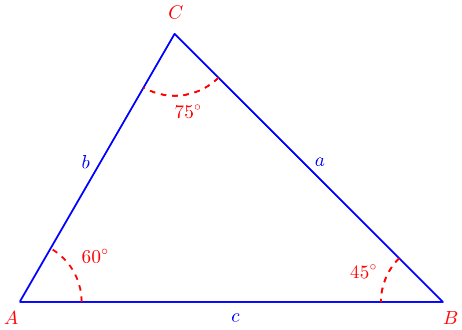
One cannot hear the shape of drum.

Theorem (Carolyn Gordon, David Webb, and Scott Wolpert, 1992)

There exists pairs of domains that are Dirichlet-isospectral (i.e. $\lambda_k(\Omega_1) = \lambda_k(\Omega_2)$ *for all* $k \geq 1$*), and also Neumann-isopectral (i.e.* $\mu_k(\Omega_1) = \mu_k(\Omega_2)$ *for all* $k \geq 1$*).*

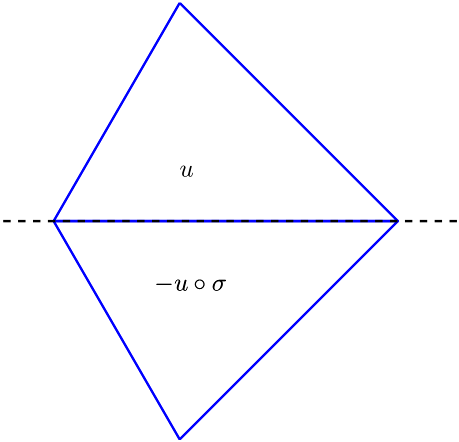
A lot of different examples have been found since then. We will study an example due to Peter Buser, John Conway, and Peter Doyle (1994), but the techniques are very general.

Building block

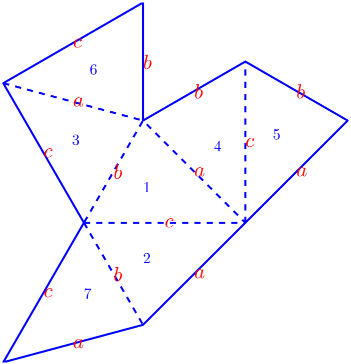


The reflexion principle

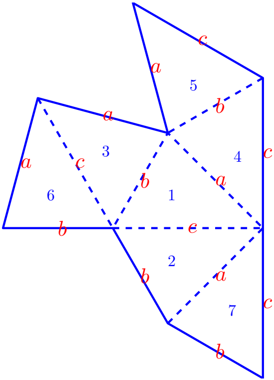
We reflect the triangle with respect to a side. If u satisfies $-\Delta u = \lambda u$ in the triangle, and a Dirichlet boundary condition on the side, the antisymmetric extension of u satisfies $-\Delta u = \lambda u$ in the symmetrized domain .



The two domains



Ω_1



Ω_2

Decompositions and transplantations

If u is a function on Ω_1, we note u_i, for $i \in \{1, \ldots, 7\}$, the restriction of u to the triangle n°i. We decompose in the same way any function v on Ω_2 We can write this

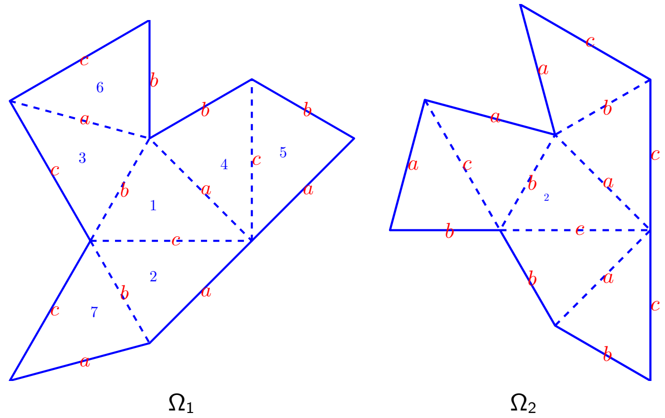
$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} \text{ and } V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{pmatrix}.$$

For all $i, j \in \{1, \ldots, 7\}$, there is a unique isometry $\tau_{i,j}$ of the triangle n°i in Ω_1 to the triangle n°j in Ω_2. There is a unique transplantation of u_i to the triangle n°j:

$$u_i \circ \tau_{i,j}^{-1}.$$

The idea of the proof is to start from an eigenfunction u in Ω_1, associated with λ, and to construct an eigenfunction v on Ω_2, associated with λ, using the transplantations of the functions u_i.

Construction of the global transplantation mapping



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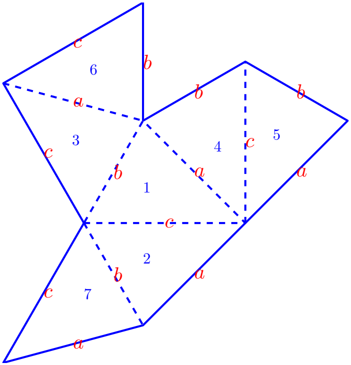

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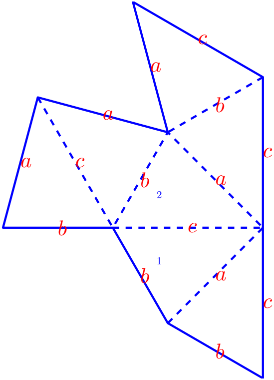
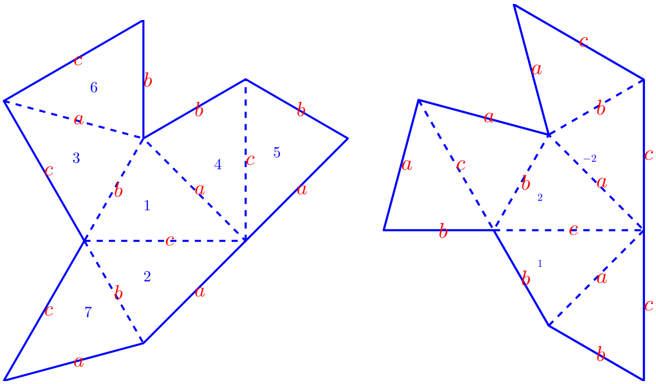


Ω_2

Construction of the global transplantation mapping



Ω_1 Ω_2

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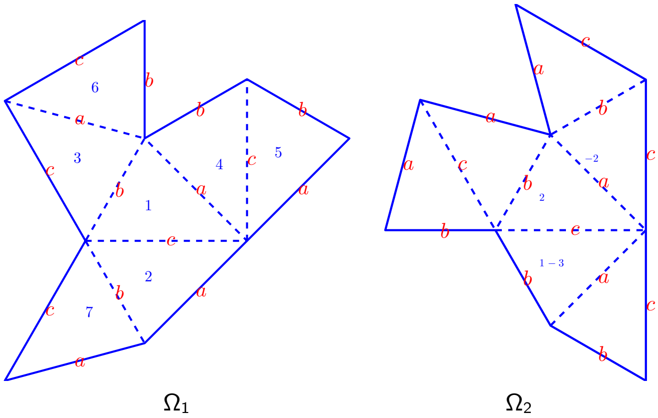

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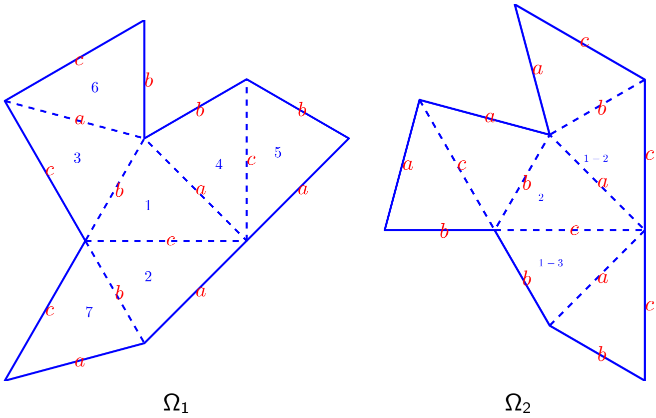

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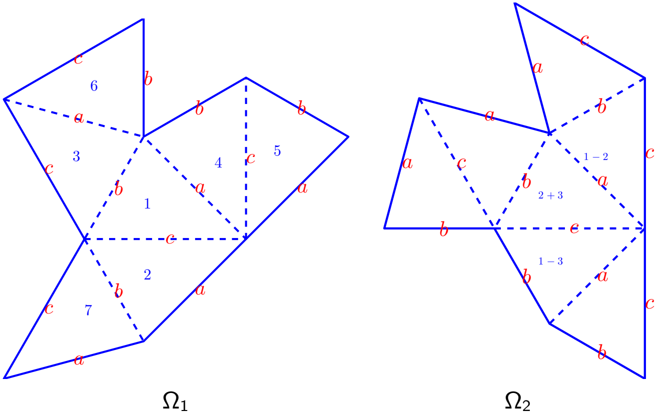

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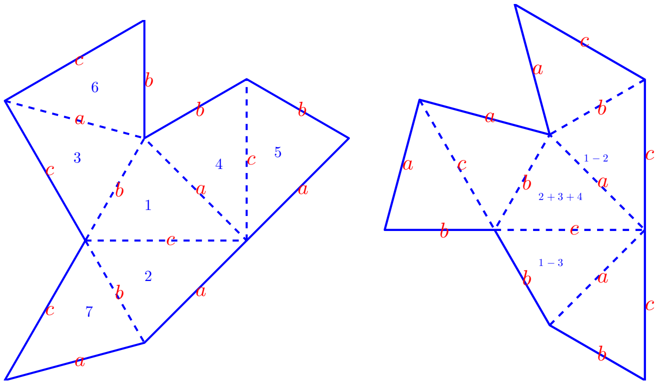

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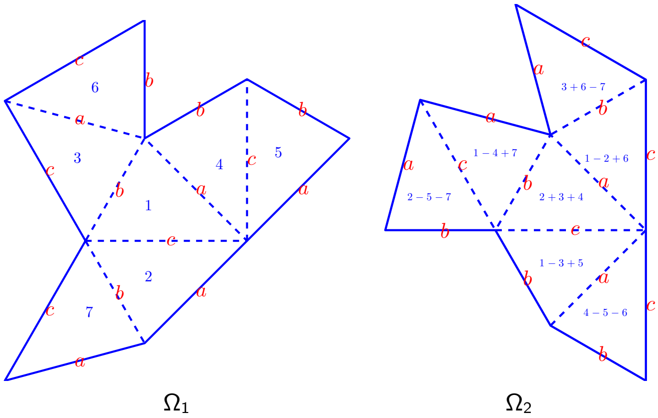

Matrix representation

The global transplantation mapping can be represented by a matrix T such that $V = TU$. Here we have:

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

This matrix is invertible, so that $U = T^{-1}V$. We find:

$$T^{-1} = \frac{1}{24}\begin{pmatrix} 4 & 8 & 8 & 8 & 4 & 4 & 4 \\ 8 & 4 & 4 & -8 & -4 & 8 & -4 \\ 8 & -8 & 4 & 4 & 8 & -4 & -4 \\ 8 & 4 & -8 & 4 & -4 & -4 & 8 \\ 4 & 8 & -4 & -4 & 4 & -8 & -8 \\ 4 & -4 & -4 & 8 & -8 & 4 & -8 \\ 4 & -4 & 8 & -4 & -8 & -8 & 4 \end{pmatrix}.$$

End of the proof

The matrix representation shows that the global transplantation mapping is an invertible linear mapping.
By construction, it sends an eigenspace for the Dirichlet eigenvalue problem in Ω_1 into an eigenspace for the Dirichlet eigenvalue problem in Ω_2.

Proposition

Any Dirichlet eigenvalue for the domain Ω_1 is also a Dirichlet eigenvalue for the domain Ω_2, with a greater or equal multiplicity.

Using the transplantation mapping defined by the matrix T^{-1}, we can exchange the role of Ω_1 and Ω_2.
We conclude that Ω_1 and Ω_2 have the same Dirichlet eigenvalues, with the same multiplicities, that is to say they are Dirichlet-isospectral.
In the same way, we could prove that Ω_1 and Ω_2 are Neumann-isospectral.

Open problems

- Are there sets of more than two isospectral domains?

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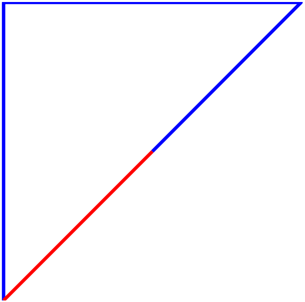
- Are there smooth isospectral domains?
- Are there simply connected isospectral domains?
- Are there convex isospectral domains?

The answer to the last question is positive if we allow mixed boundary conditions (Virginie Bonnaillie-Noël, Bernard Helffer, Thomas Hoffmann-Ostenhof, 2009). What about the Dirichlet condition?



Grazie per l'attenzione!