# Eigenvalues variations for Aharonov-Bohm operators

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# Notation

In the following, $\Omega$ denotes an open set in $\mathbb{R}^2$ with a sufficiently regular boundary (we can assume for instance that $\partial\Omega$ is piecewise $C^1$).

If $D$ is a bounded open set in $\mathbb{R}^2$, we denote by $(\lambda_k^D(D))_{k\geq 1}$ the eigenvalues of the Dirichlet Laplacian on $D$, in non-decreasing order and counted with multiplicity, i.e.

$$\begin{cases} -\Delta u &= \lambda_k^D(D)\, u \text{ in } D, \\ u &= 0 \text{ on } \partial D. \end{cases}$$

Let $\omega$ be an open set in $\mathbb{R}^2$.

- We call vector potential on $\omega$ a vector field $\mathbf{A} \in C^\infty(\omega, \mathbb{R}^2)$.
- We call magnetic Hamiltonian associated with $\mathbf{A}$ the Friedrichs's extension of the differential operator $(-i\nabla - \mathbf{A})^2$ defined on $C_c^\infty(\omega)$. We denote it by $-\Delta_{\mathbf{A}}$.
- We denote by $\mathcal{H}_{\mathbf{A}}^1$ the form domain associated with $-\Delta_{\mathbf{A}}$.

# Aharonov-Bohm Operators

- For $X_0$ in $\mathbb{R}^2$, in the polar coordinates centered at $X_0$, we define

$$\mathbf{A}_{\alpha}^{X_0}(x) = \frac{\alpha}{r}\mathbf{e}_{\theta}\,.$$

- For $\mathbf{X} = (X_1, \dots, X_N) \in \mathbb{R}^{2N}$ (assumed to be distinct for now) and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$, we define

$$\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}(x) = \sum_{i=1}^{N} \mathbf{A}_{\alpha_i}^{X_i}(x)\,.$$

- We call the magnetic Hamiltonian associated with the vector potential $\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}$ the Aharonov-Bohm operator associated with $\mathbf{X}$ and $\boldsymbol{\alpha}$. We denote it by $-\Delta_{\boldsymbol{\alpha}}^{\mathbf{X}}$ .
- We denote the form domain of $-\Delta_{\boldsymbol{\alpha}}^{\mathbf{X}}$ by $\mathcal{H}_0^1(\mathbf{X}, \boldsymbol{\alpha})$.

# Physical interpretation

- We associate with $\mathbf{A}_{\alpha}^{\mathbf{X}}$ a magnetic field (which is a measure)

$$\mathbf{B} = \mathrm{Curl}\,\mathbf{A}_{\alpha}^{\mathbf{X}} = \partial_{x_1} A_{\alpha,2}^{\mathbf{X}} - \partial_{x_2} A_{\alpha,1}^{\mathbf{X}} = \sum_{i=1}^{N} \alpha_i \delta_{X_i}\,.$$

- If $B_i$ is a small disk centered at $X_i$, such that $X_j \notin B_i$ for $j \neq i$,

$$\alpha_i = \frac{1}{2\pi} \int_{B_i} \mathbf{B}\,.$$

We call $\alpha_i$ the normalized (magnetic) flux at $X_i$.

- For or any closed loop $\gamma$ in $\mathbb{R}^2 \setminus \cup_{i=1}^{N} \{X_i\}$,

$$\oint_{\gamma} \mathbf{A}_{\alpha}^{\mathbf{X}}(s)\,ds = 2\pi \sum_{i=1}^{N} \mathrm{ind}_{\gamma}(X_i)\alpha_i\,,$$

where $\mathrm{ind}_{\gamma}(X_i)$ is the winding number of $\gamma$ around $X_i$.

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# Gauge transformations

- A gauge transformation on $\omega$ acts on pairs vector field-function as $(\mathbf{A}, u) \mapsto (\mathbf{A}^*, u^*)$, with

$$\begin{cases} \mathbf{A}^* = \mathbf{A} - i\frac{\nabla\psi}{\psi}\,, \\ u^* = \psi u\,, \end{cases}$$

  where $\psi \in C^\infty(\omega, \mathbb{C})$ satisfies $|\psi| = 1$.
- $\psi$ is called a gauge function on $\omega$.
- Two magnetic potentials are said to be gauge equivalent when the second one can be obtained from the first by a gauge transformation.
- A gauge transformation does not change the magnetic field $\mathbf{B} = \mathrm{Curl}\,\mathbf{A}$, nor the probability distribution $|u|^2$.

# Gauge invariance

## Lemma

*Let* $\mathbf{A} \in C^\infty(\omega, \mathbb{R}^2)$*. It is gauge equivalent to* $0$ *if, and only if,*

$$\frac{1}{2\pi}\oint_\gamma \mathbf{A}(s)\,ds$$

*is an integer for any loop* $\gamma$ *contained in* $\omega$ *.*

## Proposition

*If* $\mathbf{A}$ *and* $\mathbf{A}^*$ *are two gauge equivalent magnetic potentials in* $C^\infty(\omega, \mathbb{R}^2)$ *, the operators* $-\Delta_{\mathbf{A}}$ *and* $-\Delta_{\mathbf{A}^*}$ *are unitarily equivalent.*

In the case of Aharonov-Bohm operators with fixed poles, the gauge invariance allows us to remove the poles that have an integer flux. This is however not convenient in the case of moving poles, since several non-integer fluxes can add up to an integer flux.

## Form domain

### Theorem (Laptev - Weidl, 1998)

Let $X \in \mathbb{R}^2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. For every $0 < \rho$ and $u \in C_c^\infty(\mathbb{R}^2 \setminus \{X\})$,

$$\int_{B(X,\rho)} \frac{|u|^2}{|x - X|^2}\, dx \leq \mathrm{dist}(\alpha, \mathbb{Z})^{-2} \int_{B(X,\rho)} \left|(-i\nabla - \mathbf{A}_\alpha^X)\, u\right|^2 dx\,.$$

In the case where there is no integer flux, the preceding inequality shows that $\mathcal{H}_0^1(\mathbf{X}, \boldsymbol{\alpha})$ is embedded in $H^1(\Omega)$. In the general case, we prove the following result with the help of some local gauge transformations.

### Proposition

*The form domain $\mathcal{H}_0^1(\mathbf{X}, \boldsymbol{\alpha})$ is compactly embedded in $L^2(\Omega)$.*

# Statement of the problem

- Since the form domain $\mathcal{H}_0^1(\mathbf{X},\boldsymbol{\alpha})$ is compactly embedded in $L^2(\Omega)$, the operator $-\Delta_{\boldsymbol{\alpha}}^{\mathbf{X}}$ has compact resolvent. Thus, its spectrum consists in isolated eigenvalues, each with finite multiplicity.
- We denote by $(\lambda_k(\mathbf{X},\boldsymbol{\alpha}))_{k\geq 1}$ the sequence of these eigenvalues, arranged in non-decreasing order and counted with multiplicity.
- We define (for each $k$) the function $\mathbf{X}\mapsto\lambda_k(\mathbf{X},\boldsymbol{\alpha})$ on $\mathbb{R}^{2N}$. For instance, we set $\lambda_k((X,X,Y),(\alpha,\beta,\gamma))=\lambda_k((X,Y),(\alpha+\beta,\gamma))$.
- For fixed $k$ and $\boldsymbol{\alpha}$, we study the regularity of the function $\mathbf{X}\mapsto\lambda_k(\mathbf{X},\boldsymbol{\alpha})$.

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# Continuity of the eigenvalues

## Theorem

*For all integer $k \geq 1$ and all $\boldsymbol{\alpha} \in \mathbb{R}^N$, the function $\mathbf{X} \mapsto \lambda_k(\mathbf{X}, \boldsymbol{\alpha})$ is continuous on $\mathbb{R}^{2N}$.*

## Corollary

*Let us assume that $\Omega$ that is simply connected, that $\alpha \in \mathbb{R}$, that $X_0 \in \partial\Omega$, and that $X \in \Omega$. Then*

$$\lim_{X \to X_0} \lambda_k(X, \alpha) = \lambda_k^D(\Omega).$$

## Corollary

*Let us assume that $\Omega$ is simply connected, $X_0 \in \Omega$ and that $X$, $Y \in \Omega$. Then*

$$\lim_{X, Y \to X_0} \lambda_k((X, Y), (1/2, 1/2)) = \lambda_k^D(\Omega).$$

# Tools

## Proposition (characterization of the form domain)

We have

$$\mathcal{H}_0^1(\mathbf{X},\alpha)=\left\{u\in L^2(\Omega)\,;\,\left(-i\nabla-\mathbf{A}_\alpha^{\mathbf{X}}\right)u\in L^2(\Omega)\text{ et }\gamma_0 u=0\right\}\,,$$

where $\gamma_0$ is the usual boundary trace operator, of possibly its conjugate by a gauge transformation.

## Lemma (non-concentration inequality)

There exists a constant $C\geq 0$ such that, for any $x_0\in\Omega$, $r>0$, and $u\in\mathcal{H}_0^1(\mathbf{X},\alpha)$,

$$\|u\|^2_{L^2(B(x_0,r))}\leq C\,r\left(\|u\|^2_{L^2(\mathbb{R}^2)}+\|(-i\nabla-\mathbf{A}_\alpha^{\mathbf{X}})u\|^2_{L^2(\mathbb{R}^2)}\right)\,.$$

Let $(\mathbf{X}^{(n)})_{n\geq 1}$ be a sequence of elements of $\mathbb{R}^{2N}$ such that $\mathbf{X}^{(n)} \to \mathbf{X}$ . We want to prove that for all $k \geq 1$, $\lambda_k(\mathbf{X}^{(n)}, \boldsymbol{\alpha}) \to \lambda_k(\mathbf{X}, \boldsymbol{\alpha})$ .

## Lemma (limsup)

For each $k \geq 1$, $\limsup_{n\to+\infty} \lambda_k(\mathbf{X}^{(n)}, \boldsymbol{\alpha}) \leq \lambda_k(\mathbf{X}, \boldsymbol{\alpha})$.

According to the min-max formula

$$\lambda_k(\mathbf{X}, \alpha) = \inf_{\varphi_1,\dots,\varphi_k \in C_c^\infty(\Omega_\mathbf{X})} \max_{u \in vect(\varphi_1,\dots,\varphi_k)} \frac{\|(-i\nabla - \mathbf{A}_\alpha^\mathbf{X})u\|^2}{\|u\|^2} .$$

We choose $\varphi_1, \dots, \varphi_k \in C_c^\infty(\Omega_\mathbf{X})$ such that

$$\max_{u \in vect(\varphi_1,\dots,\varphi_k)} \frac{\|(-i\nabla - \mathbf{A}_\alpha^\mathbf{X})u\|^2}{\|u\|^2} \leq \lambda_k(\mathbf{X}, \boldsymbol{\alpha}) + \varepsilon.$$

Then, we can show

$$\limsup_{n\to+\infty} \lambda_k(\mathbf{X}^{(n)}, \boldsymbol{\alpha}) \leq \limsup_{n\to+\infty} \max_{u \in vect(\varphi_1,\dots,\varphi_k)} \frac{\|(-i\nabla - \mathbf{A}_\alpha^{\mathbf{X}^{(n)}})u\|^2}{\|u\|^2} \leq \lambda_k(\mathbf{X}, \boldsymbol{\alpha}) + \varepsilon.$$

## Lemma (extraction)

Let $(\lambda^{(n)})$ be a bounded sequence of eigenvalues of $-\Delta_{\alpha}^{\mathbf{X}^{(n)}}$ and $(u^{(n)})$ a corresponding sequence of normalized eigenfunctions. There exist $\lambda \in \mathbb{R}$ and $u \in \mathcal{H}_0^1(\mathbf{X}, \alpha)$ such that, for some subsequence,

i. $\lambda^{(n_p)} \to \lambda$ ,
ii. $u^{(n_p)} \to u$ in $L^2(\Omega)$ ,
iii. $u$ is an eigenfunction of $-\Delta_{\alpha}^{\mathbf{X}}$ associated with $\lambda$.

We can now prove that $\lambda_1(\mathbf{X}^{(n)}, \alpha) \to \lambda_1(\mathbf{X}, \alpha)$. According to the limsup Lemma, $\limsup_{n\to+\infty} \lambda_1(\mathbf{X}^{(n)}, \alpha) \leq \lambda_1(\mathbf{X}, \alpha)$. This implies that $(\lambda_1(\mathbf{X}^{(n)}, \alpha))$ is bounded. But then, any limit point of $(\lambda_1(\mathbf{X}^{(n)}, \alpha))$ is an eigenvalue of $-\Delta_{\alpha}^{\mathbf{X}}$, and therefore $\lambda_1(\mathbf{X}, \alpha) \leq \liminf_{n\to+\infty} \lambda_1(\mathbf{X}^{(n)}, \alpha))$.

We treat the case $k > 1$ by induction.

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# Analyticity of the eigenvalues

We note $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ the vectors of the canonical basis.

**Theorem**

*Let $\mathbf{X} \in \mathbb{R}^{2N}$, and $k \geq 1$ an integer, such that $X_i \notin \partial\Omega$ for all $1 \leq i \leq N$, $X_i \neq X_j$ for all $i \neq j$, and $\lambda_k(\mathbf{X}, \alpha)$ is a simple eigenvalue. The function*

$$(t_1, t_2, \dots, t_{2N-1}, t_{2N}) \mapsto \lambda_k((X_1 + t_1\mathbf{e}_1 + t_2\mathbf{e}_2, \dots, X_N + t_{2N-1}\mathbf{e}_1 + t_{2N}\mathbf{e}_2), \alpha)$$

*is then analytic in a neighborhood of 0.*

- In the case of one pole, a translation show that the problem is equivalent to moving the boundary, and we can use the Hadamard theory.
- If $N \geq 2$, we cannot use translations to fix the poles. Instead, we construct a suitable family of diffeomorphisms to rewrite the problem as a perturbation of an Aharonov-Bohm operator with fixed poles.
- We then show that this perturbation is relatively bounded and apply the Kato-Rellich theory.

# $K_{\mathbf{X}}$-real functions

Let us study the case where $\alpha_i \in \mathbb{Z} + 1/2$ for all $1 \le i \le N$. According to the condition on circulations, $2\mathbf{A}_\alpha^{\mathbf{X}}$ is gauge equivalent to $0$. There exists a gauge function $\psi$ defined on $\Omega_{\mathbf{X}}$ such that $-i\overline{\psi}\nabla\psi = 2\mathbf{A}_\alpha^{\mathbf{X}}$. We fix such a $\psi$. We define $K_{\mathbf{X}}$ by $K_{\mathbf{X}}u = \psi\,\overline{u}$. We say that a function $u$ is $K_{\mathbf{X}}$-real if $K_{\mathbf{X}}u = u$.

A direct calculation shows that $-\Delta_\alpha^{\mathbf{X}} \circ K_{\mathbf{X}} = K_{\mathbf{X}} \circ (-\Delta_\alpha^{\mathbf{X}})$. We can therefore choose a basis of $K_{\mathbf{X}}$-real eigenfunctions for $-\Delta_\alpha^{\mathbf{X}}$.

**Theorem (Alziary - Fleckinger-Pellé - Takáč, 2003)**

Let $u$ be a $K_{\mathbf{X}}$-real eigenfunction associated with $\lambda_k(\mathbf{X},\alpha)$. Let $(r,\theta)$ be the polar coordinates centered at $X_i$ in a neighborhood of $X_i$. There exist a non-negative integer $m$ and $C^1$-functions $f$ and $g$ such that $f(0) \neq 0$,

$$u(x) = r^{m+1/2} f(x) \quad \text{and} \quad (-i\nabla - \mathbf{A}_\alpha^{\mathbf{X}}(x))u(x) = r^{m-1/2} g(x).$$

Furthermore, $2m+1$ is the number of nodal lines meeting at $X_i$.

# Critical points for the eigenvalues

**Theorem**

*Let $\mathbf{X} \in \mathbb{R}^{2N}$. We set $\alpha = (1/2, \ldots, 1/2) \in \mathbb{R}^N$. Let us fix $1 \leq i \leq N$, with $X_i \in \Omega$ and $\mathbf{v} \in \mathbb{R}^2$. For all $k \geq 1$ and $t \in \mathbb{R}$, we define*

$$\mathbf{X}(t) = (X_1, \ldots, X_i + t\mathbf{v}, \ldots, X_N)$$

*and $\lambda_k(t) = \lambda_k(\mathbf{X}(t), \alpha)$. Let us assume that the eigenvalue $\lambda_k(0)$ is simple, and admits a $K_{\mathbf{X}}$-real eigenfunction with at least three nodal lines that meet at $X_i$. Then $\lambda_k'(0) = 0$.*

To prove this, we construct a family of diffeomorphisms $\Phi_{h,t}$ that depends on the additional parameter $h > 0$. Using the Feynman-Hellmann formula, we compute $\lambda_k'(0)$ (which does not depend on $h$) as an integral $I(h)$ depending on $h$ (integral of a function supported on a disk of size $h$ centered at $X_i$). We then use the local estimates on $u$, with $m \geq 1$, to show that $\lim_{h \to 0} I(h) = 0$.

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# The problem of minimal partitions

- A $k$-partition $\mathcal{P}$ of $\Omega$, with $k$ a positive integer, is a set $\mathcal{P} = \{D_1, \ldots, D_k\}$ of $k$ disjoint, connected, and open sets contained in $\Omega$.
- The energy of a given $k$-partition $\mathcal{D}$ is defined by

$$\Lambda_k(\mathcal{D}) = \max_{1 \leq i \leq k} \lambda_1^D(D_i)$$

- Let us note

$$\mathfrak{L}_k(\Omega) = \inf \left\{ \Lambda_k(\mathcal{D})\,;\ \mathcal{D} = \{D_1, \ldots, D_k\}\ k\text{-partition of } \Omega \right\}\,.$$

- The $k$-partition $\mathcal{D}$ is called minimal if

$$\mathfrak{L}_k(\Omega) = \max_{1 \leq i \leq k} \lambda_1^D(D_i)\,.$$

# Magnetic characterization of minimal partitions

**Theorem (Helffer - Hoffmann-Ostenhof, 2013)**

Let us assume that the open set $\Omega$ is diffeomorphic to a disk with a finite number of holes. Let us assume that $\mathcal{D} = \{D_1, \ldots, D_k\}$ is a minimal $k$-partition of $\Omega$. There exist a finite number of points $X_1, \ldots, X_N$ in $\mathbb{R}^2$ such that $\mathcal{D}$ is the nodal partition associated with a $K_{\mathbf{X}}$-real eigenfunction $u$ of the operator $-\Delta_{\alpha}^{\mathbf{X}}$, with

$$\mathbf{X} = (X_1 \ldots, X_N)$$

and

$$\alpha = (1/2, \ldots, 1/2).$$

Furthermore, the eigenfunction $u$ is associated with the eigenvalue $\lambda_k(\mathbf{X}, \alpha)$.

To build the magnetic potential, we have to add poles :

- at each singular point of the boundary of $\mathcal{D}$ where an odd number of curves meet,
- in each hole with an odd number of curves touching its boundary.

## Theorem

*Let us assume that* $\Omega$ *is a connected open set,* $k$ *a positive integer, and* $\mathcal{P}$ *a minimal* $k$*-partition of* $\Omega$ *. We denote by* $\mathbf{X} = (X_1 \, , \, \dots \, , \, X_N)$ *and* $\alpha = (1/2 \;\; \dots, \, 1/2)$ *poles and fluxes as defined in the magnetic characterization. Let us additionally assume that the eigenvalue* $\lambda_k(\mathbf{X}, \alpha)$ *is simple. The point* $\mathbf{X}$ *is then critical for the function* $\mathbf{Y} \mapsto \lambda_k(\mathbf{Y}, \alpha)$ *, which is defined and analytic in a neighborhood of* $\mathbf{X}$ *.*

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# Example : circular sector

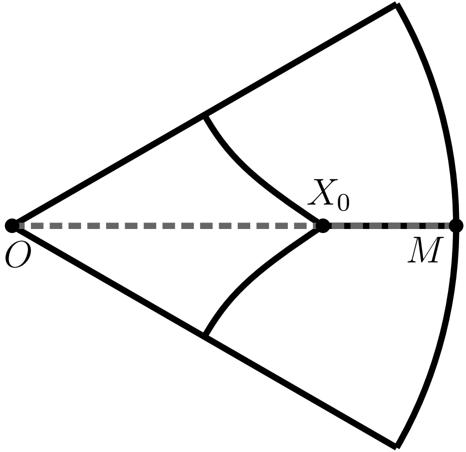


Figure: Aharonov-Bohm problem with symmetry.

# Search for a 3-partition (V. Bonnaillie-Noël)

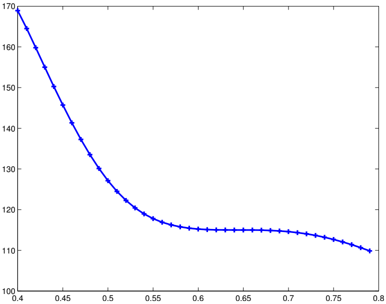


Figure: Eigenvalue $\lambda_3(X, 1/2)$ as a function of $x$.

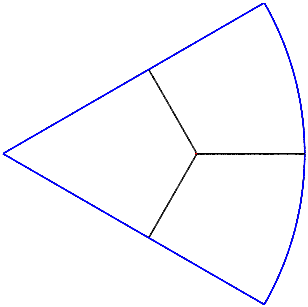

Figure: Nodal set of a third eigenfunction of an Aharonov-Bohm operator.

There is a point of inflexion for $x \simeq 0.64$, that corresponds to three nodal lines meeting at $X = (x, 0)$.