The Courant-sharp property for flat tori

Calculus of variations and PDEs

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Plan

1. Review of general results
2. The square two-dimensional torus
3. The cubic three-dimensional torus

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Introduction

- M compact connected manifold (with or without boundary) of dimension n ;
- $\lambda_1(M) < \lambda_2(M) \leq \dots \leq \lambda_k(M) \leq \dots$ sequence of the eigenvalues of $-\Delta_M$, with Dirichlet condition on ∂M, counted with multiplicities.

If u eigenfunction :

- nodal set : $\mathcal{N}(u) := \overline{\{x \in M\,;\ u(x) = 0\}}$;
- nodal domain : connected component of $M \setminus \mathcal{N}(u)$;
- $\nu(u)$ number of nodal domains.

If λ eigenvalue, $\kappa(\lambda) := \min\{k \geq 1\,;\ \lambda = \lambda_k(M)\}$.

Courant theorem

If u is an eigenfunction of $-\Delta_M$ associated with the eigenvalue λ,

$$\nu(u) \leq \kappa(\lambda).$$

Refinements of Courant theorem

Definition

The eigenvalue λ is called Courant-sharp if there exists an associated eigenfunction u such that $\nu(u) = \kappa(\lambda)$.

Remark : $\lambda_1(M)$ and $\lambda_2(M)$ are always Courant-sharp.

Theorem (Pleijel, 1956)

If Ω is a bounded open set in $\mathbb{R}^2$ with a sufficiently regular boundary, only a finite number of eigenvalues $(\lambda_k(\Omega))_{k\geq 1}$ are Courant-sharp.

Set $\nu_k := \max\{\nu(u)\,;\ u \text{ associated with } \lambda_k(\Omega)\}$. Then $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2} < 1$.

Theorem (Bérard–Meyer, 1982)

For all $n \geq 2$, there exists $\gamma_n < 1$ such that, for all compact manifold of dimension n, with or without boundary,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma_n.$$

Application : minimal partitions

- k-partition : family of k open, connected and disjoint subsets in M, $\mathcal{D} = \{D_1, \ldots, D_k\}$.
- Energy : $\Lambda_k(\mathcal{D}) := \max_{1 \leq i \leq k} \lambda_1(D_i)$.
- $\mathfrak{L}_k(M) := \inf_{\mathcal{D}} \Lambda_k(\mathcal{D})$.
- Minimal k-partition $\mathcal{D}^*$: $\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(M)$ (Existence and regularity : Bucur–Buttazzo–Henrot, 1998 ; Conti–Terracini–Verzini, 2005 ; Caffarelli–Lin, 2007 ; Helffer–Hoffmann-Ostenhof–Terracini, 2009).
- Nodal partition : family of the nodal domains of an eigenfunction of $-\Delta_M$.
- A nodal partition associated with a Courant-sharp eigenvalue is minimal.
- The converse is true when $n = 2$: if the partition associated with (λ, u) is minimal then $\nu(u) = \kappa(\lambda)$ (Helffer–Hoffmann-Ostenhof–Terracini, 2009).
- Pleijel theorem : for a given domain in $\mathbb{R}^2$, minimal k-partitions are not nodal, except for of finite number of k's.

Proof of Pleijel result

- Let u, associated with $\lambda_k(\Omega)$, have ν_k nodal domains $D_1, \dots, D_{\nu_k}$.
- Applying Faber-Krahn : $\lambda_k(\Omega) = \lambda_1(D_i) \geq \frac{\pi j_{0,1}^2}{|D_i|}$ pour $1 \leq i \leq \nu_k$.
- Summing : $\nu_k \pi j_{0,1}^2 \leq \lambda_k(\Omega)|\Omega|$.
- Weyl's law : $\lambda_k(\Omega) \sim \frac{4\pi k}{|\Omega|}$.
- Conclusion : $\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j_{0,1}^2} \simeq 0.69.$

Remarks :

- Bérard-Meyer : $\gamma_n = \frac{(2\pi)^n}{\omega_n j^n_{(n-1)/2,1}} = \frac{2^{n-2} n^2 \Gamma(\frac{n}{2})^2}{j^2_{(n-2)/2,1}} \leq \gamma_2 = \frac{4}{j_{0,1}^2}$.
- Pleijel's result holds for eigenfunctions of the Neumann Laplacian on domains in $\mathbb{R}^2$ having a piecewise analytic boundary (Polterovich, 2009).
- Conjecture (Polterovich, 2009) : $\limsup_{k\to+\infty} \frac{\nu_k}{k} = \frac{2}{\pi} \simeq 0.64$ (sharp for a rectangle).

The case of manifolds

Use of the co-area formula and symmetrization of the level sets of a positive eigenfunction to prove a Faber-Krahn type inequality from an isoperimetric inequality.

Without any condition on the volume, the isoperimetric ratio for a domain in a compact manifold without boundary can be arbitrarily small.

Bérard and Meyer proved asymptotic inequalities for domains with small volume.

- For all $\varepsilon > 0$, there exists $V(\varepsilon, M) > 0$ such that, if $|D| \leq V(\varepsilon, M)$, $|\partial D||D|^{-\frac{n-1}{n}} \geq (1-\varepsilon)|\partial B||B|^{-\frac{n-1}{n}}$.
- If $|D| \leq V(\varepsilon, M)$, $\lambda_1(D)|D|^{\frac{2}{n}} \geq (1-\varepsilon)^2\lambda_1(B)|B|^{\frac{2}{n}}$.

Some examples

All Courant-sharp eigenvalues besides $\lambda_1(M)$ and $\lambda_2(M)$ are known in some specific examples.

- Square, Dirichlet case (λ_4) (Pleijel, 1956 ; Bérard–Helffer 2014) ;
- Sphere $\mathbb{S}^2$ (none) (Leydold, 1996) ;
- Disk, Dirichlet case (λ_4) (Helffer–Hoffmann-Ostenhof–Terracini, 2009) ;
- Square, Neumann case (λ_4, λ_5, and λ_9) (Helffer–Persson-Sundqvist, 2014) ;
- Square torus (none) (L., 2014) ;
- Equilateral torus (none), equilateral (λ_4), hemi-equilateral (none) and right angled isosceles (none) triangles (Bérard–Helffer, 2015) ;
- Disk, Neumann case (λ_4), $\mathbb{S}^{d-1}$ for $d \geq 4$ (none), and unit ball in dimension $d \geq 3$, Dirichlet and Neumann cases (none) (Helffer–Persson-Sundqvist, 2015) ;
- Cube, Dirichlet case (none) (Helffer–Kiwan, 2015) ;
- Right-angled isosceles triangle, Neumann case (λ_3, λ_4 and λ_6) (Band–Bersudsky–Fajman, 2015)
- Cubical torus (none) (L. ; 2015).

Plan

Presentation of the example

$\mathbb{T}^2$ flat square torus of dimension 2 : $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$.

Eigenvalues of $-\Delta_{\mathbb{T}^2}$: $\lambda_{m,n} = 4\pi^2(m^2 + n^2)$.

Eigenfunctions :

$$u^{cc}_{m,n}(x,y) = \cos(2m\pi x)\cos(2n\pi y);$$
$$u^{cs}_{m,n}(x,y) = \cos(2m\pi x)\sin(2n\pi y);$$
$$u^{sc}_{m,n}(x,y) = \sin(2m\pi x)\cos(2n\pi y);$$
$$u^{ss}_{m,n}(x,y) = \sin(2m\pi x)\sin(2n\pi y).$$

Vector space of eigenfunctions $E_{m,n}$, of dimension 1, 2 or 4.

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n)\in\mathbb{N}^2} E_{m,n}}.$$

$\lambda_1(\mathbb{T}^2) = \lambda_{0,0} = 0$ and $\lambda_k(\mathbb{T}^2) = \lambda_{1,0} = \lambda_{0,1} = 4\pi^2$ for $k \in \{2,3,4,5\}$.

Statement of the result

Theorem

The only Courant-sharp eigenvalues of $-\Delta_{\mathbb{T}^2}$ are $\lambda_k(\mathbb{T}^2)$ for $k \in \{1, 2, 3, 4, 5\}$ (first and second eigenvalues).

Corollary

The minimal k-partitions of $\mathbb{T}^2$ *are nodal only for* $k \in \{1, 2\}$.

Corollary

For $k \geq 3$*, we have* $\nu_k \leq k - 1$.

Inequalities

Theorem (Howards–Hutchings–Morgan, 1999)

Let $\mathbb{T}$ be a flat torus of dimension 2 whose closed geodesics have minimal length a. Let $0 < A < |\mathbb{T}|$. The region of $\mathbb{T}$ having area A and minimal perimeter is

- a (round) disk if $0 < A \leq \frac{a^2}{\pi}$;
- a strip bounded by geodesics if $\frac{a^2}{\pi} \leq A \leq |\mathbb{T}| - \frac{a^2}{\pi}$;
- the complement of a disk if $|\mathbb{T}| - \frac{a^2}{\pi} \leq A < |\mathbb{T}|$.

Proposition

Let $D \subset \mathbb{T}^2$ such that $|D| \leq \frac{1}{\pi}$. Then $\lambda_1(D)|D| \geq \pi j_{0,1}^2$.

Upper and lower bounds

Lemma

If λ is a Courant-sharp eigenvalue with $\kappa(\lambda) \geq 4$, then $\kappa(\lambda) \leq \frac{\lambda}{\pi j_{0,1}^2}$.

Proof : for u associated eigenfunction with $\kappa(\lambda)$ nodal domains, there is one nodal domain D satisfying $|D| \leq \frac{1}{\kappa(\lambda)} < \frac{1}{\pi}$, and therefore $\pi j_{0,1}^2 \leq \lambda_1(D)|D| \leq \frac{\lambda}{\kappa(\lambda)}$.

$N(\lambda) := \sharp\{k \,:\, \lambda_k(\mathbb{T}^2) < \lambda\}$ (counting function).

Lower bound : $N(\lambda) > \frac{\lambda}{4\pi} - \frac{2\sqrt{\lambda}}{\pi} - 3$.

For an eigenvalue λ, $\kappa(\lambda) = N(\lambda) + 1$.

A priori bound : λ is not Courant-sharp if $\kappa(\lambda) \geq 50$.

Reduction to a finite list

$\frac{\lambda}{4\pi^2}$	indices	multiplicity	κ	$\frac{\lambda}{\pi j_{0,1}^2}$
0	$(0,0)$	1	1	
1	$(1,0)$, $(0,1)$	4	2	
2	$(1,1)$	4	6	4.35
4	$(2,0)$, $(0,2)$	4	10	8.69
5	$(2,1)$, $(1,2)$	8	14	10.86
8	$(2,2)$	4	22	17.38
9	$(3,0)$, $(0,3)$	4	26	19.56
10	$(3,1)$, $(1,3)$	8	30	21.73
13	$(3,2)$, $(2,3)$	8	38	28.25
16	$(4,0)$, $(0,4)$	4	46	34.77

TABLE: The 49 first eigenvalues

Plan

Presentation of the example

$\mathbb{T}^3$ the flat cubic torus of dimension 3 : $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$.

Eigenvalues of $-\Delta_{\mathbb{T}^3}$: $\lambda_{m,n,p} = 4\pi^2(m^2 + n^2 + p^2)$.

Eigenfunctions :

$$u_{m,n,p}(x,y,z) = \varphi(2m\pi x)\psi(2n\pi y)\chi(2p\pi z),$$

with φ, ψ, and χ in $\{\cos, \sin\}$.

Vector space of eigenfunctions $E_{m,n,p}$, of dimension 1, 2, 4 or 8.

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n,p)\in\mathbb{N}^2} E_{m,n,p}}.$$

$\lambda_1(\mathbb{T}^3) = \lambda_{0,0,0} = 0$ and $\lambda_k(\mathbb{T}^3) = \lambda_{1,0,0} = \lambda_{0,1,0} = \lambda_{0,0,1} = 4\pi^2$ for $k \in \{2,3,4,5,6,7\}$.

Statement of the result

Theorem

The only Courant-sharp eigenvalues of $-\Delta_{\mathbb{T}^3}$ are $\lambda_k(\mathbb{T}^3)$ for $k \in \{1, 2, 3, 4, 5, 6, 7\}$ (first and second eigenvalues).

Corollary

For $k \geq 3$, we have $\nu_k \leq k - 1$.

Isoperimetric inequality

Main difficulty : the isoperimetric problem on the torus is not solved in dimension 3 or larger.

There are partial results for the periodic isoperimetric problems.

Theorem (Hauswirth–Perez–Romon–Ros, 2004)

Let $\mathcal{U} \subset \mathbb{T}^2 \times \mathbb{R}$ with $|\mathcal{U}| \leq \frac{4\pi}{81}$. Then

$$\left|\partial B^3\right| \left|B^3\right|^{-\frac{2}{3}} \leq |\partial \mathcal{U}| \, |\mathcal{U}|^{-\frac{2}{3}}.$$

Spheres-cylinders-planes profile

For $V \in (0, +\infty)$, $I(V) := \inf \left\{ |\partial\Omega| \ : \ \Omega \subset \mathbb{T}^2 \times \mathbb{R} \text{ and } |\Omega| = V \right\}$.

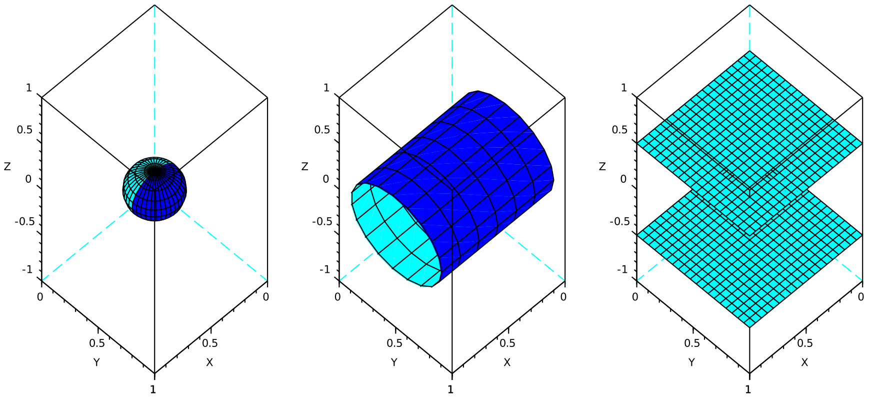
Minimizing among regions bounded by spheres, cylinders and pairs of two-dimensional planar tori produces the spheres-cylinders-planes profile . For $V \in (0, 1]$,

$$I_{SCP}(V) = \begin{cases} (36\pi)^{1/3} V^{2/3} & \text{if } 0 < V \leq \frac{4\pi}{81} & \text{(sphere);} \\ 2\pi^{1/2} V^{1/2} & \text{if } \frac{4\pi}{81} \leq V \leq \frac{1}{\pi} & \text{(cylinder);} \\ 2 & \text{if } \frac{1}{\pi} \leq V & \text{(pair of tori).} \end{cases}$$

Conjecture : $I = I_{SCP}$.

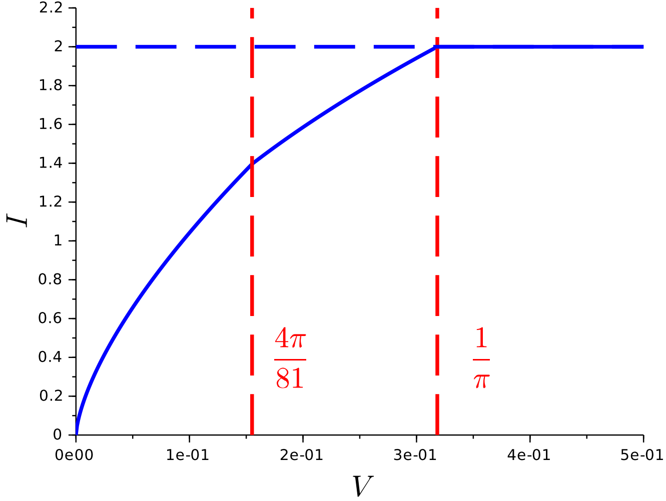
Previous result : $I = I_{SCP}$ in the spherical range.

Conjectured isoperimetric domains for $\mathbb{T}^2 \times \mathbb{R}$



(a) $0 < V \leq \frac{4\pi}{81}$ (b) $\frac{4\pi}{81} \leq V \leq \frac{1}{\pi}$ (c) $\frac{1}{\pi} \leq V$

Conjectured isoperimetric profile $\mathbb{T}^2 \times \mathbb{R}$



Inequalities in $\mathbb{T}^3$

Proposition

Let Ω be an open set in $\mathbb{T}^3$ with $|\Omega| \leq \frac{4\pi}{81}$. We have

$$\left|\partial B^3\right| \left|B^3\right|^{-\frac{2}{3}} \leq (|\partial\Omega| + 2\,|\Omega|)\,|\Omega|^{-\frac{2}{3}}\ .$$

Restatement : $\left(1-\left(\frac{2|\Omega|}{9\pi}\right)^{\frac{1}{3}}\right)\left|\partial B^3\right|\left|B^3\right|^{-\frac{2}{3}} \leq |\partial\Omega|\,|\Omega|^{-\frac{2}{3}}$.

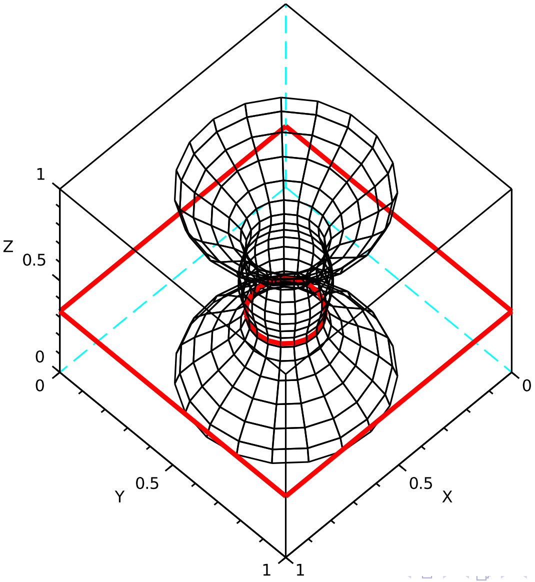
Corollary

Let Ω be an open set in $\mathbb{T}^3$ with $|\Omega| \leq \frac{4\pi}{81}$, we have

$$\left(1-\left(\frac{2\,|\Omega|}{9\pi}\right)^{\frac{1}{3}}\right)^2 \lambda_1(B^3)\left|B^3\right|^{\frac{2}{3}} \leq \lambda_1(\Omega)\,|\Omega|^{\frac{2}{3}}\,.$$

Remark : $\lambda_1(B^3) = \pi^2$.

Cutting procedure (adapted from Bérard–Meyer, 1982)



Proof of the isoperimetric inequality

Isoperimetric inequality :

- $\mathcal{H}_{z=t} := \left\{(x,y,z) \in \mathbb{T}^3 \ : \ z = t\right\}$.
- $|\Omega| = \int_0^1 |\Omega \cap \mathcal{H}_{z=t}| \ dt$.
- There exists $t_z \in (0,1)$ such that $|\Omega \cap \mathcal{H}_{z=t_z}| \leq |\Omega|$.
- We consider $\widetilde{\Omega} := \Omega \setminus \mathcal{H}_{z=t}$ as a subset of $\mathbb{T}^2 \times \mathbb{R}$; we have $|\widetilde{\Omega}| = |\Omega|$ and $|\partial\widetilde{\Omega}| \leq |\partial\Omega| + 2|\Omega|$.
- We apply the isoperimetric inequality in $\mathbb{T}^2 \times \mathbb{R}$.

Upper and lower bounds

Lemma

If λ *is a Courant-sharp eigenvalue of* $-\Delta_{\mathbb{T}^3}$ *with* $\kappa(\lambda) \geq 7$ *, then*

$$\kappa(\lambda) \leq \left(\left(\frac{3}{4\pi^4} \right)^{\frac{1}{3}} \sqrt{\lambda} + \left(\frac{2}{9\pi} \right)^{\frac{1}{3}} \right)^3 .$$

Proposition

For $\lambda \geq 12\pi^2$, $N(\lambda) > \frac{4\pi}{3} \left(\frac{\sqrt{\lambda}}{2\pi} - \sqrt{3} \right)^3$.

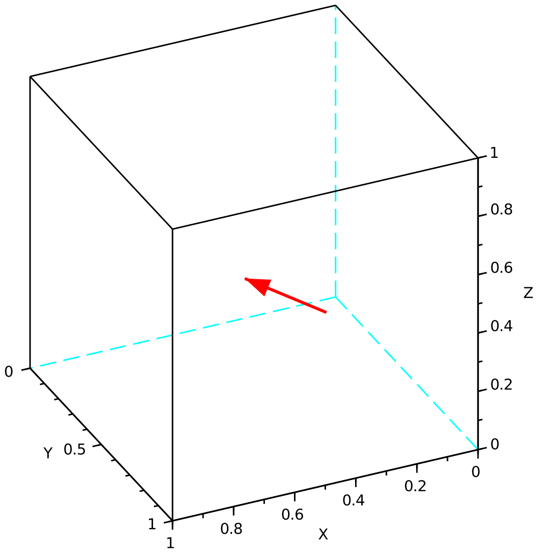
Reduction to a finite list

A priori bound : λ is a not Courant-sharp eigenvalue if $\kappa(\lambda) \geq 1378$.

After examination of the remaining eigenvalues, the only ones which can be Courant-sharp are :

- $\lambda_1(\mathbb{T}^3) = \lambda_{0,0,0} = 0$;
- for $k \in \{2, \dots, 7\}$, $\lambda_k(\mathbb{T}^3) = \lambda_{1,0,0} = \lambda_{0,1,0} = \lambda_{0,0,1} = 4\pi^2$;
- for $k \in \{8, \dots, 19\}$, $\lambda_k(\mathbb{T}^3) = \lambda_{1,1,0} = \lambda_{1,0,1} = \lambda_{0,1,1} = 8\pi^2$.

An isometry of $\mathbb{T}^3$



Courant Theorem with symmetry

- Isometry : $\sigma(x,y,z) := (x+1/2, y+1/2, z+1/2)$.
- Space of symmetric functions : $\mathcal{S} := \{u \in L^2(\mathbb{T}^3) \,:\, u\circ\sigma = u\}$.
- H_S restriction of $-\Delta_{\mathbb{T}^3}$ to $\mathcal{S}$.
- $(\lambda_k^S)_{k\geq 1}$ spectrum of $H_{\mathcal{S}}$.
- If λ eigenvalue of H_S, $\kappa_S(\lambda) := \inf\{k \,:\, \lambda_k^S = \lambda\}$

If u is an eigenfunction of H_S, and D a nodal domain of u, then $\sigma(D)$ is also a nodal domain of u. Either $\sigma(D) = D$: the domain is symmetric, or $\sigma(D) \neq D$: $\{D, \sigma(D)\}$ is a pair of isometric domains. We denote by $\alpha(u)$ the number of symmetric domains and by $\beta(u)$ the number of pairs. We have $\nu(u) = \alpha(u) + 2\beta(u)$.

Courant theorem with symmetry (Leydold, 1996 ; Helffer–Hoffmann-Ostenhof–Terracini, 2010)

If u is an eigenfunction of H_S associated with the eigenvalue λ,

$$\alpha(u) + \beta(u) \leq \kappa_S(\lambda).$$

Conclusion

Remark :

$$u_{m,n,p}(x+1/2, y+1/2, z+1/2) = \varphi(2m\pi x + m\pi)\psi(2n\pi y + n\pi)\chi(2p\pi z + p\pi) = (-1)^{m+n+p}u_{m,n,p}(x,y,z) = (-1)^{m^2+n^2+p^2}u_{m,n,p}(x,y,z)$$

Consequence : if u is an eigenfunction associated with the eigenvalue $8\pi^2$, it is symmetric , and $\nu(u) = \alpha(u) + 2\beta(u) \leq 2(\alpha(u) + \beta(u))$.

On the other hand, $8\pi^2$ is an eigenvalue of H_S with $\kappa_S(8\pi^2) = 2$. According to Courant theorem with symmetry, $\alpha(u) + \beta(u) \leq 2$.

Therefore, $\nu(u) \leq 4$ (sharp bound) while $\kappa(8\pi^2) = 8$.

The eigenvalue $8\pi^2$ of $-\Delta_{\mathbb{T}^3}$ is not Courant-sharp.