# Examples of spectral minimal partitions

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# Statement of the problem

- Setting: $\Omega$ bounded open set in $\mathbb{R}^2$ or in a 2-dimensional Riemannian manifold.
- For $D$ subset of $\Omega$, $(\lambda_k(D))_{k\geq 1}$ eigenvalues of the Dirichlet-Laplacian in $D$.
- $k$-partition of $\Omega$: family $\mathcal{D} = (D_1, \dots, D_k)$ of $k$ open, connected and mutually disjoint subsets of $\Omega$.
- Energy: $\Lambda_k(\mathcal{D}) = \max_{1\leq i\leq k} \lambda_1(D_i)$.
- A $k$-partition $\mathcal{D}^*$ is called minimal if

$$\Lambda_k(\mathcal{D}^*) = \mathfrak{L}_k(\Omega),$$

with

$$\mathfrak{L}_k(\Omega) = \inf\left\{\Lambda_k(\mathcal{D}); \mathcal{D} \text{ } k\text{-partition of } \Omega\right\}.$$

# Existence and regularity

Existence and regularity follow from the work of several authors :Bucur, Buttazzo, and Henrot; Caffarelli and Lin; Conti, Terracini, and Verzini; Helffer, Hoffmann-Ostenhof, and Terracini. We refer to the results by Helffer, Hoffmann-Ostenhof, and Terracini (2009).

## Assumption A

$\Omega$ has a piecewise-$C^{1,+}$ boundary and satisfies the interior cone property.

## Theorem (existence)

Under assumption A, for any positive integer $k$, there exists a minimal $k$-partition of $\Omega$.

## Theorem (regularity)

Under assumption A, for any positive integer $k$, any minimal $k$-partition of $\Omega$ is regular up to 0-capacity sets.

# More on regularity

We say that the $k$-partition $\mathcal{D} = (D_1, \dots, D_k)$ is strong if

$$\Omega = \mathrm{Int}\left(\cup_{i=1}\overline{D_i}\right) \setminus \partial\Omega.$$

In that case we define the boundary of $\mathcal{D}$ as $N(\mathcal{D}) := \overline{\cup_{i=1}^{k}\partial D_i \setminus \partial\Omega}$.
We say that $\mathcal{D}$ is regular if it is strong and its boundary $N := N(\mathcal{D})$ satisfies the following properties.

- i. $\Omega \cap N$ is locally a $\mathcal{C}^{1,1-}$, curve except for the points in a finite set $S_{int}$;
- ii. to each $x \in S_{int}$ corresponds an integer $\nu(x) \geq 2$ such that $N$, in a neighborhood of $x$, consists of $v(x)$ half-curves of class $\mathcal{C}^{1,+}$ ending at $x$;
- iii. $S_{bd} = \partial\Omega \cap N$ is finite and to each $z \in S_{bd}$ corresponds an integer $\rho(z) \geq 1$ such that, in a neighborhood of $z$, $N$ consists of $\rho(z)$ half-curves of class $\mathcal{C}^{1,+}$ contained in $\overline{\Omega}$ et and meeting $\partial\Omega$ at $z$;
- iv. at each point in $S_{int}$, the half-curves make equal angles;
- v. at each point in $N \cap \partial\Omega$, the half-curves and $\partial\Omega$ make equal angles.

Points iv. and v. will be called the equal angle meeting property.

# Nodal partitions

If $u$ is an eigenfunction of the Dirichlet Laplacian in $\Omega$, the connected components of the complement of its zero set are called its nodal domains. Let us denote by $\nu(u)$ the number of nodal domain of $u$. The family $\mathcal{D}_u = (D_i)_{1\leq i\leq \nu(u)}$ of all the nodal domains of $u$ is the nodal partition associated with $u$.

Given a regular $k$-partition $\mathcal{D} = (D_i)_{1\leq i\leq k}$, we say that two domains $D_i$ and $D_j$ are neighbors if the set

$$D_{i,j} := \mathrm{Int}\left(\overline{D}_i \cup \overline{D}_j\right)$$

is connected.

### Theorem

A minimal $k$-partition of $\Omega$ is nodal if, and only if, it is bipartite, that is to say if one can color its domains with only two colors such that two neighbors have a different color.

### Courant Theorem

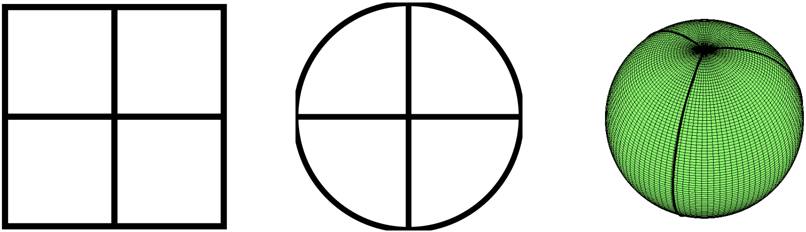
If $u$ is an eigenfunction associated with $\lambda_k(\Omega)$, then $\nu(u) \leq k$.

### Theorem (Courant-sharp characterization)

The nodal partition associated with the eigenfunction $u$ is minimal if, and only if, $u$ is Courant-sharp, that is to say associated with $\lambda_k(\Omega)$, where $k = \nu(u)$.

In particular, a minimal 2-partition is always the nodal partition associated with a second eigenfunction.

# Examples of minimal $k$-partitions



(a) $k = 4$, nodal. (b) $k = 4$, nodal. (c) $k = 3$, non-nodal.

# Magnetic characterization of minimal partitions I

We define the elementary magnetic potential

$$\mathbf{A}_0(r\cos\theta, r\sin\theta) = \frac{1}{2r}(-\sin\theta, \cos\theta).$$

For $\mathbf{X} = (X_1, \dots, X_N) \in \mathbb{R}^2$, we define the multipolar magnetic potential

$$\mathbf{A}_{\mathbf{X}}(x) = \sum_{j=1}^{N} \mathbf{A}_0(x - X_j)$$

and the multipolar Aharonov-Bohm Hamiltonian

$$H_{\mathbf{X}} = \left(-i\nabla - \mathbf{A}_{\mathbf{X}}(x)\right)^2,$$

with Dirichlet boundary condition.
We denote by $(\lambda_k^{AB}(\mathbf{X}))_{k\geq 1}$ the associated eigenvalues.

# Magnetic characterization of minimal partitions II

### Theorem (Helffer and Hoffmann-Ostenhof, 2013)

Let us assume that $\mathcal{D} = \{D_1, \dots, D_k\}$ is a minimal $k$-partition of $\Omega \subset \mathbb{R}^2$. There exist a finite number of points $X_1, \dots, X_N$ in $\mathbb{R}^2$ such that $\mathcal{D}$ is the nodal partition associated with an eigenfunction $u$ of the operator $H_{\mathbf{X}}$, with

$$\mathbf{X} = (X_1 \dots, X_N)$$

Furthermore, the eigenfunction $u$ is associated with the eigenvalue $\lambda_k^{AB}(\mathbf{X})$ .

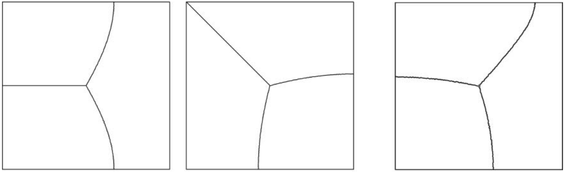
To build the magnetic potential, we have to add poles:

- at each singular point of the boundary of $\mathcal{D}$ where an odd number of curves meet,
- in each hole with an odd number of curves touching its boundary.

# Examples

From Bonnaillie-Noël, Helffer, and Hoffmann-Ostenhof. We have

$$\Lambda_3(\mathcal{D}) = \lambda_3^{AB}(C) \simeq 66.581.$$

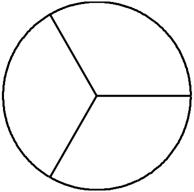


(a) Candidates with different symmetries.

(b) Asymmetric candidate.

For the unit disk, we have the Mercedes star conjecture (Bonnaillie-Noël, Helffer, and Hoffmann-Ostenhof):

$$\Lambda_3(\mathcal{D}) = \lambda_3^{AB}(C) \simeq 20.1907.$$



# Sub-partition property

### Theorem

Let $\mathcal{D} = (D_i)_{1 \leq i \leq k}$ be a minimal $k$-partition of $\Omega$. Let $I \subset \{1, \ldots, k\}$ with $k' := \sharp I$, $k' < k$, such that

$$\Omega_I := \mathrm{Int}\left(\bigcup_{i \in I} \overline{\mathcal{D}}_i\right)$$

is a connected open set. Then the sub-partition $\mathcal{D}_I = (\mathcal{D})_{i \in I}$ is the unique minimal $k'$-partition of $\Omega_I$ (up to 0-capacity sets).

### Corollary (pair compatibility condition)

Let $\mathcal{D} = (D_i)_{1 \leq i \leq k}$ $(k \geq 3)$ be a minimal $k$-partition of $\Omega$. For any two neighbors $D_i$ and $D_i$, the second eigenvalue of the Dirichlet Laplacian on

$$D_{i,j} := \mathrm{Int}\left(\overline{D}_i \cup \overline{D}_j\right),$$

is simple, and $D_i$ and $D_j$ are the nodal domains of any eigenfunction associated with $\lambda_2(D_{i,j})$.

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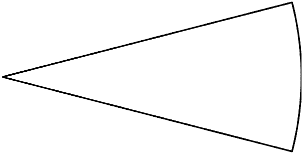
# Setting

Circular sector of opening $\alpha \in (0, 2\pi]$:

$$\Sigma_\alpha = \left\{ (\rho \cos\theta, \rho \sin\theta) \quad : \; 0 < \rho < 1 \text{ and } -\tfrac{\alpha}{2} < \theta < \tfrac{\alpha}{2} \right\}$$



Eigenvalues:

$$\lambda_{m,n}(\alpha) = j^2_{m\frac{\pi}{\alpha},n}$$

Orthogonal basis of eigenfunctions:

$$u^\alpha_{m,n}(\rho,\theta) = J_{m\frac{\pi}{\alpha}}(j_{m\frac{\pi}{\alpha},n}\rho) \sin(m\pi(\tfrac{\theta}{\alpha} + \tfrac{1}{2}))$$

Here $j_{m\frac{\pi}{\alpha},n}$ is the $n$-th positive zero of the Bessel function of the first kind $J_{m\frac{\pi}{\alpha}}$.

# Nodal minimal partitions for small $\alpha$

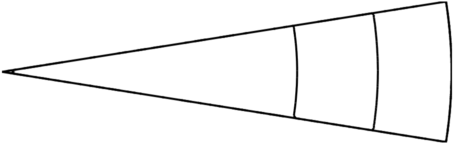
## Proposition

Let $k \geq 2$, we define $\alpha_k^1 := \inf\{\alpha \in (0, 2\pi] \ : \ \lambda_{1,k}(\alpha) \geq \lambda_{2,1}(\alpha)\}$. Then $\alpha_k^1 > 0$ and for any $\alpha \in (0, \alpha_k^1)$, there is a unique minimal $k$-partition of $\Sigma_\alpha$, which is nodal, and more precisely consists of the nodal domains of $u_{1,k}^\alpha$.

Proof:

$$\lambda_{m,n}(\alpha) = \frac{m^2\pi^2}{\alpha^2} + 2^{2/3}|a_n| \left(\frac{m\pi}{\alpha}\right)^{4/3} + \mathcal{O}\left(\alpha^{-2/3}\right).$$

Here $a_n$ is the $n$-th negative zero of the Airy function $Ai$.



# Nodal minimal partitions for large $\alpha$

## Proposition

For $2 \leq k \leq 5$, we define

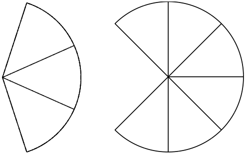
$$\alpha_k^2 = \inf\{\alpha \in (0, 2\pi] : \lambda_{k,1}(\alpha) < \lambda_{1,2}(\alpha)\}.$$

Then for any $\alpha \in [\alpha_k^2, 2\pi]$, the nodal partition associated with $u_{k,1}^{\alpha}$ is a minimal $k$-partition, that is to say

$$\mathfrak{L}_k(\Sigma_\alpha) = \lambda_{k,1}(\alpha).$$

For $2 \leq k \leq 5$ and $\alpha \in (\alpha_k^2, 2\pi]$, the minimal $k$-partition therefore consists of $k$ equal angular sectors.

For $k \geq 6$, any $k$-partition consisting of $k$ angular sectors is not minimal.



# Summary for $k = 3$

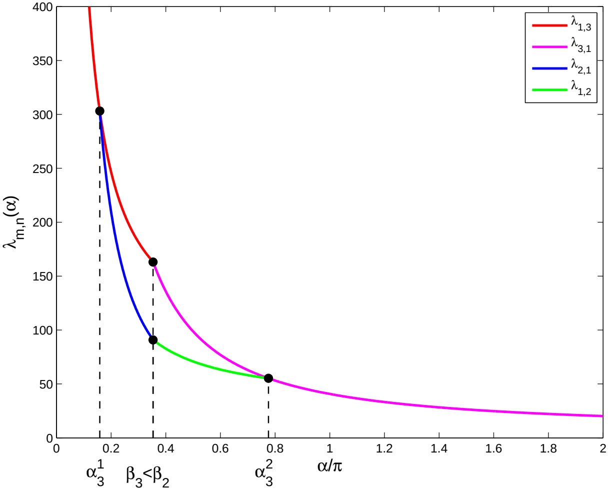
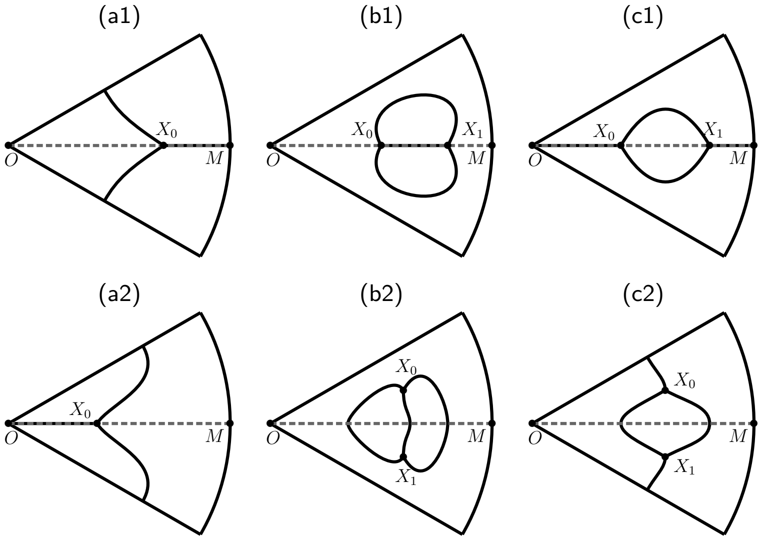


Figure: Upper and lower bounds for $\mathfrak{L}_3(\Sigma_\alpha)$.

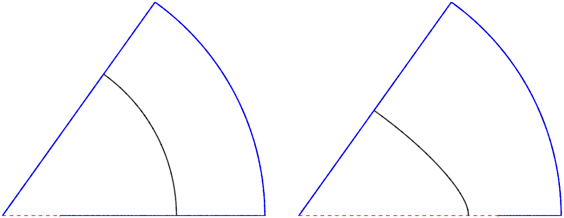
# Possible types of symmetric non-bipartite 3-partitions



Mixed Neumann-Dirichlet problem on the upper-half sector $\Sigma_\alpha^+$ associated with (a1):

$$\begin{cases} -\Delta\varphi = \lambda_2^{ND}(x_\alpha)\varphi & \text{in } \Sigma_\alpha^+, \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [O, X_0^\alpha], \\ \varphi = 0 & \text{elsewhere.} \end{cases}$$

We vary $X_0^\alpha = (x_\alpha, 0)$ until we obtain a 3-partition whose energy cannot be decreased.



$y^{ND}(x_\alpha) > x_\alpha$
3-partition

$y^{ND}(x_\alpha) < x_\alpha$
2-partition

We obtain in the way $L_3^{sym}(\alpha)$, the infimum of the energy among symmetric 3-partitions.

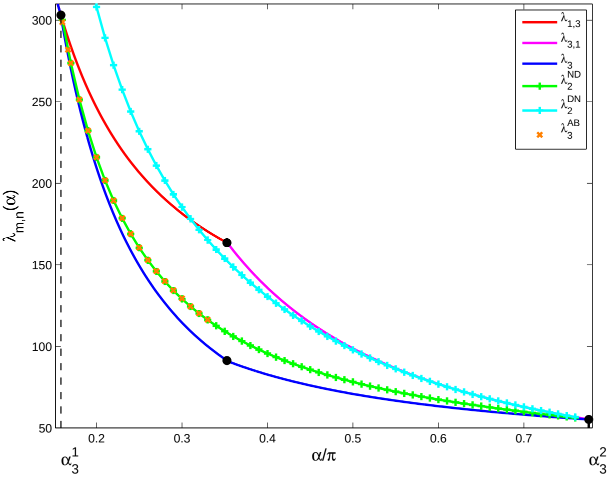
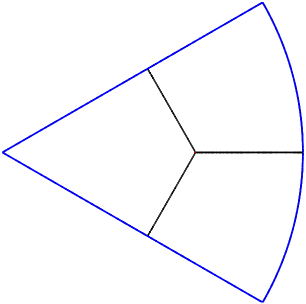
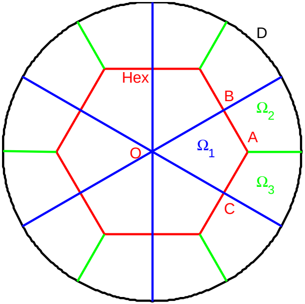


Figure: New upper bound $\mathfrak{L}_3(\Sigma_\alpha)$.

# Conjecture for $\alpha = \pi/3$





It seems that the boundary consists of straight lines and that the domains have equal area, which would imply

$$x_{\pi/3} = \frac{1}{3}\sqrt{\frac{2\sqrt{3}\pi}{3}} \simeq 0.634875\ldots$$

# Non-symmetric candidates

For $X \in \Sigma_\alpha$, we compute numerically the eigenvalues $\lambda_k^{AB}(X)$ of the Aharonov-Bohm operator $(-i\nabla - \mathbf{A}_X)^2$, by a double covering approach.
We note that for $\alpha = \alpha_3^1 + \varepsilon$, with $\varepsilon > 0$ small enough,

$$\lambda_2^{ND}(x_\alpha) > \lambda_3^{AB}(X_0^\alpha, 1/2).$$

This implies that the symmetric 3-partition constructed before is not minimal.
To obtain a better, non-symmetric, candidate, we start from a non-symmetric nodal partition of $\Sigma_{\alpha_3^1}$ having a singular point on the boundary. We increase $\alpha$, we push the singular point $X$ inside the domain, and we compute $\lambda_3^{AB}(X)$ and an associated eigenfunction.

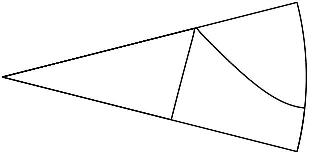
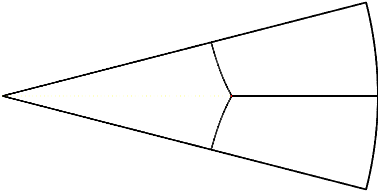
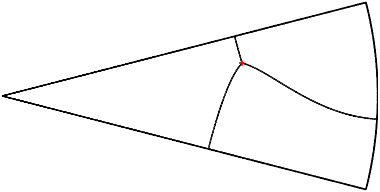


Figure: Minimal 3-partition for $\alpha = \alpha_3^1$.

# Results for $\alpha = \frac{16\pi}{100}$



$L_3^{sym}(\frac{16\pi}{100}) = 300.190$



$\lambda_3^{AB}(X) = 298.787$

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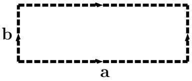
# Setting

Flat torus of length $a$ and width $b$: $\mathrm{T}(a,b) = (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$.



Eigenvalues:

$$\lambda_{m,n}(a,b) = 4\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Orthogonal basis of eigenfunctions:

$$u_{m,n}^{a,b}(x,y) = \varphi\left(\frac{2m\pi x}{a}\right) \psi\left(\frac{2n\pi y}{b}\right),$$

with $\varphi, \psi \in \{\cos, \sin\}$.
Partition of $\mathrm{T}(a,b)$ into $k$ equal vertical strip: $\mathcal{D}_k(a,b) = (D_i)_{1 \le i \le k}$, with

$$D_i = \left( \frac{i-1}{k} a, \frac{i}{k} a \right) \times (0,b).$$

It has energy $\Lambda_k(\mathcal{D}_k(a,b)) = k^2\pi^2/a^2$. 

# Minimality of $\mathcal{D}_k(1, b)$

Transition value:

$$b_k = \sup\{b \in (0,1] \ ; \ \mathcal{D}_k(1,b) \text{ is a minimal } k\text{-partition of } \mathsf{T}(1,b)\}.$$

**Proposition**

The partition $\mathcal{D}_k(1, b)$ is minimal for all $b \in (0, b_k]$.

**Theorem (B. Helffer and T. Hoffmann-Ostenhof)**

If $k$ is even , $b_k = 2/k$.
If $k$ is odd, $b_k \geq 1/k$.

**Proposition (V. Bonnaillie-Noël, C. L.)**

If $k$ is odd, $b_k \geq b_k^{\mathsf{S}} > 1/k$.

**Conjecture**

If $k$ is odd, $b_k = 2/\sqrt{k^2 - 1}$.

# More on the transition value for $k$ odd

For $b \in (0,1]$, we consider the infinite strip $\mathsf{S}_b = \mathbb{R} \times (0,b)$ and we define

$$b_k^{\mathsf{S}} = \sup \left\{ b \in (0,1] \,;\, j(b) > k^2\pi^2 \right\},$$

with

$$j(b) = \inf_{\Omega \subset \mathsf{S}_b, |\Omega| \le b} \lambda_1(\Omega).$$

## Proposition

$$\frac{1}{k} < b_k^{\mathsf{S}} < \frac{1}{\sqrt{k^2-1}}.$$

## Problem 1

Find an explicit lower bound of $b_k^{\mathsf{S}}$ (as a function of $k$).

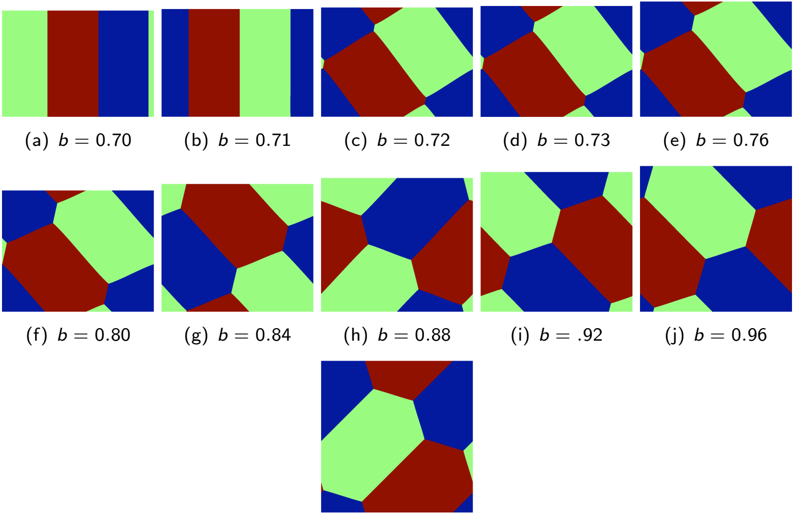
# Numerical method (based on Bourdin, Bucur, and Oudet, 2009)

- Approximation of the max by an $\ell_p$-norm: $\Lambda_{k,p}(\mathcal{D}) = \left(\frac{1}{k}\sum_{i=1}^{k}\lambda_1(D_i)^p\right)^{\frac{1}{p}}$ with $p \in [1,\infty)$;
- Relaxation of the eigenvalue problem and penalization: for $\varepsilon > 0$ and $f : \mathrm{T}(1,b) \to [0,1]$, we define $\lambda_1(f,\varepsilon)$ as the first eigenfunction of $-\Delta + \frac{1}{\varepsilon}(1-f)$, i.e.

$$\lambda_1(f,\varepsilon) = \inf_{u \in H_0^1(\mathrm{T}(1,b))\setminus\{0\}} \frac{\int_{\mathrm{T}(1,b)}(|\nabla u|^2 + \frac{1}{\varepsilon}(1-f)u^2)\,dxdy}{\int_{\mathrm{T}(1,b)}|u|^2\,dxdy}.$$

- Replacement of $k$-partitions by $k$-uples of functions $(f_1,\dots,f_k)$ satisfying $\sum_{i=1}^{k} f_i \equiv 1$.
- Discretization of the problem by a five points finite difference method for the Laplacian.
- Optimization by projected gradient.

# 3-partitions

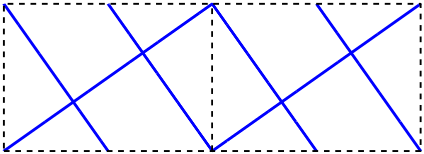


(a) $b = 0.70$ (b) $b = 0.71$ (c) $b = 0.72$ (d) $b = 0.73$ (e) $b = 0.76$

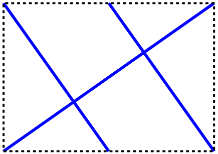
(f) $b = 0.80$ (g) $b = 0.84$ (h) $b = 0.88$ (i) $b = .92$ (j) $b = 0.96$

(k) $b = 1$

# Behavior at $b = \frac{1}{\sqrt{2}}$



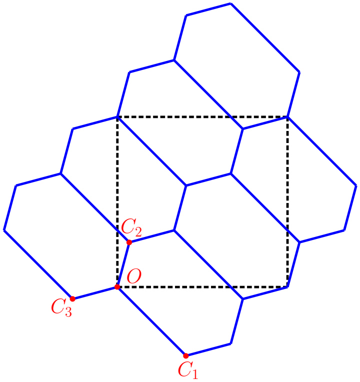
(l) A nodal 6-partition of $\mathrm{T}(2, 1/\sqrt{2})$ (associated with $\cos(3\pi x) - \cos(\pi x)\cos(2\sqrt{2}\,\pi y) + \sin(\pi x)\sin(2\sqrt{2}\,\pi y)$).



(m) The 3-partition of $\mathrm{T}(1, 1/\sqrt{2})$ after projection.

Conjectured deformation mechanism: splitting of the singular points of order 4 into pairs of singular points of order 3.

# Hexagonal tilings



If we compute $\lambda_1$ for the tilling domain by a finite element method, we get a good upper bound for $\mathfrak{L}_3(T(1,b))$.

# Upper bounds from hexagonal tilings

## Proposition

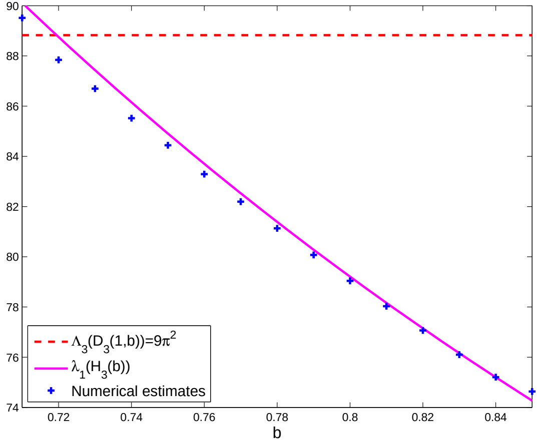
For $k \in \{3\,,\,4\,,\,5\}$, there exists $b_k^{\mathrm{H}} \in (0,1)$ such that, for any $b \in (b_k^{\mathrm{H}}, 1]$, there exists a tiling of $\mathrm{T}(1,b)$ by $k$ hexagons that satisfies the equal angle meeting property. We denote by $\mathrm{H}_k(b)$ the corresponding tiling domain, and we have

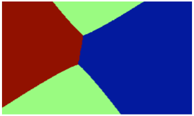
$$\mathfrak{L}_k(\mathrm{T}(1,b)) \leq \min\Big(k^2\pi^2, \lambda_1(\mathrm{H}_k(b))\Big)\,, \quad \forall b \in (b_k^{\mathrm{H}}, 1].$$

More explicitly, we can choose

$$\begin{array}{rcll} b_3^{\mathrm{H}} & = & \frac{\sqrt{11}-\sqrt{3}}{4} & \simeq & 0.396, \\ b_4^{\mathrm{H}} & = & \frac{1}{2\sqrt{3}} & \simeq & 0.289, \\ b_5^{\mathrm{H}} & = & \frac{\sqrt{291}-5\sqrt{3}}{36} & \simeq & 0.233. \end{array}$$

# Behavior near $b = \frac{1}{\sqrt{2}}$





Remark: the equal angle meeting property should imply that the boundary of the partition is curved in the neighborhood of the critical points.

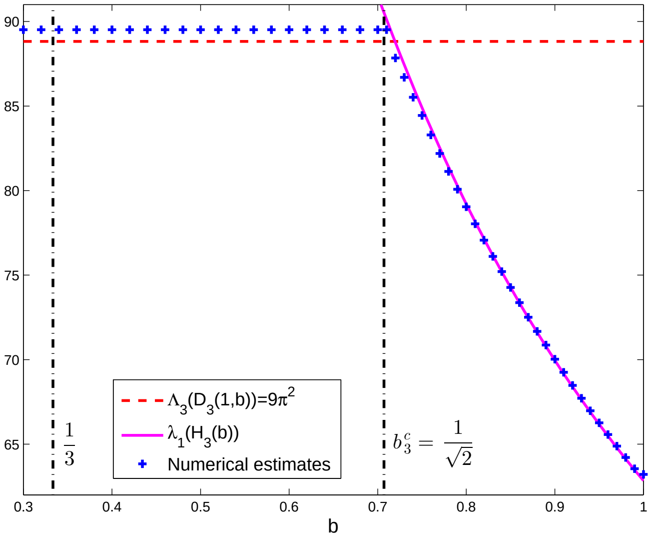
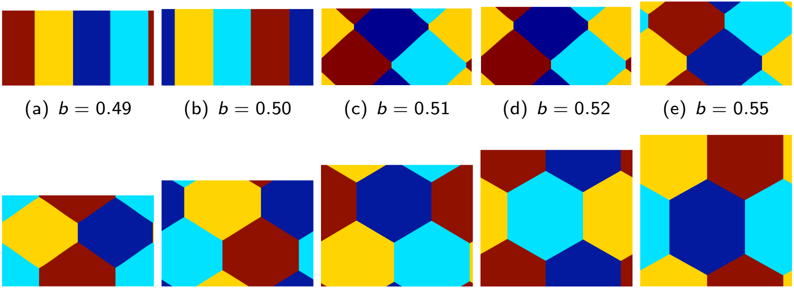


Figure: Upper bound of $\mathfrak{L}_3(T(1,b))$ as a function of $b$

# 4-partitions



(a) $b = 0.49$ (b) $b = 0.50$ (c) $b = 0.51$ (d) $b = 0.52$ (e) $b = 0.55$

(f) $b = 0.60$ (g) $b = 0.7$ (h) $b = 0.8$ (i) $b = 0.9$ (j) $b = 1$

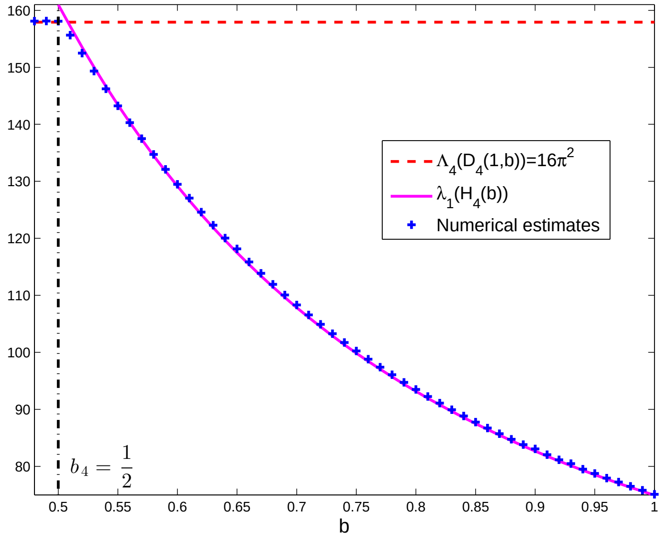
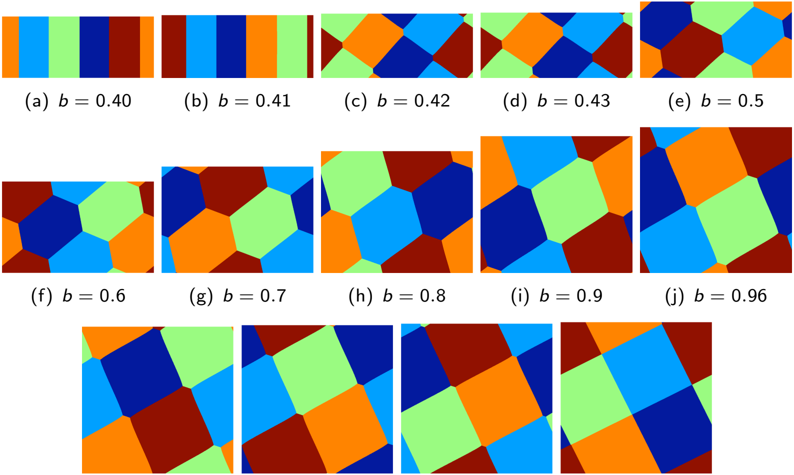


Figure: Upper bound of $\mathfrak{L}_4(T(1,b))$ as a function of $b$

# 5-partitions



(a) $b = 0.40$ (b) $b = 0.41$ (c) $b = 0.42$ (d) $b = 0.43$ (e) $b = 0.5$

(f) $b = 0.6$ (g) $b = 0.7$ (h) $b = 0.8$ (i) $b = 0.9$ (j) $b = 0.96$

(k) $b = 0.97$ (l) $b = 0.98$ (m) $b = 0.99$ (n) $b = 1$

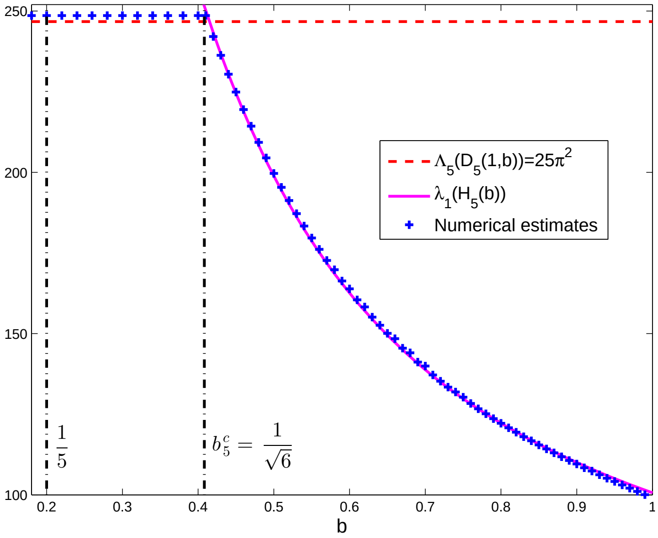
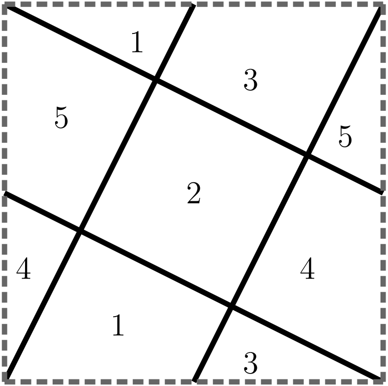


Figure: Upper bound of $\mathfrak{L}_5(T(1,b))$ as a function of $b$

# Partition of $T(1,1)$ into 5 squares



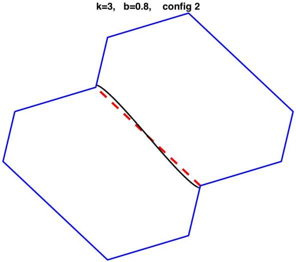
Energy: $\Lambda_5 = 10\pi^2 \simeq 98.6960 < \lambda(\mathsf{H}_5(1)) \simeq 100.6450$.

# Verification of the pair compatibility condition I

If the hexagonal tiling of $T(1, b)$ by $\mathrm{H}_k(b)$ is minimal, then for all pairs of tiling domains

$$\lambda_1(\mathrm{H}_k(b)) = \lambda_2(2\mathrm{H}_k^j(b)) \quad \text{for } j \in \{1, 2, 3\}.$$

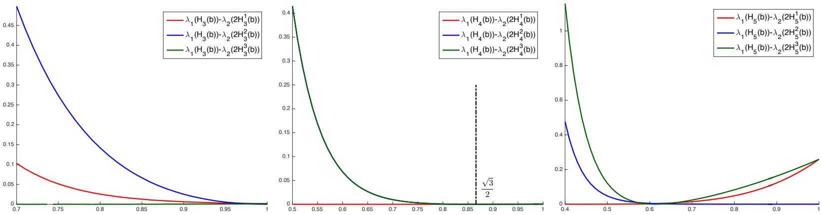
Typically, this is not satisfied.



Example for $k = 3$ and $b = 0.8$:

$$\lambda_1(\mathrm{H}_3(0.8)) \simeq 79.2072 > 79.0653 \simeq \lambda_2(2\mathrm{H}_3^2(0.8)).$$

# Verification of the pair compatibility condition II



(a) $k = 3$ (b) $k = 4$ (c) $k = 5$

Figure: $b \mapsto \delta_k^j(b)$, $j \in \{1,\ 2,\ 3\}$, $b \in (b_k^{\mathrm{H}}, 1]$ for $k = 3, 4, 5$.

In the cases $k \in \{3, 4\}$ for $b = 1$, our computations are not precise enough to conclude.

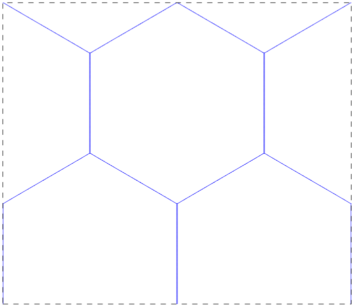
## Problem 2

Find a way to compute the eigenvalues with high accuracy.

# Special cases

The pair compatibility condition is satisfied by symmetry in the following cases:

- when $k = 5$ and $b = 1$ for the square partition;
- when $k = 4$ and $b = \frac{\sqrt{3}}{2}$ (in that case the tiling hexagon is regular).



# Plan

Review of minimal partitions

Circular sectors

Flat tori

Some open questions

# The hexagonal conjecture I

Given a domain $\Omega \subset \mathbb{R}^2$, what is the behavior of $\mathfrak{L}_k(\Omega)$ as $k \to +\infty$?

Hexagonal conjecture for the maximum (van den Berg, Helffer, Hoffmann-Ostenhof, and Terracini)

Asymptotically, a minimal partition of $\Omega$ is hexagonal, and therefore

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)|\Omega|}{k} = \lambda_1(\mathsf{H}),$$

where H is the regular hexagon of area 1.

It is easy to see that

$$\limsup_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)|\Omega|}{k} \leq \lambda_1(\mathsf{H}) \simeq 18.5901,$$

and by Faber-Krahn inequality

$$\limsup_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)|\Omega|}{k} \geq \pi j^2 \simeq 18.1684.$$

# The hexagonal conjecture II

We define

$$\mathfrak{L}_{k,1}(\Omega) = \inf\left\{ \frac{1}{k}\sum_{i=1}^{k} \lambda_1(D_i)\,;\, \mathcal{D} = (D)_{1\leq i\leq k} \ k\text{-partition of } \Omega \right\}.$$

A 1-minimal $k$-partition of $\Omega$ realizes this infimum. We have immediately

$$\mathfrak{L}_{k,1}(\Omega) \leq \mathfrak{L}_k(\Omega).$$

The following conjecture is therefore stronger that the previous one.

Hexagonal conjecture for the sum (Caffarelli and Lin, 2007)

Asymptotically, a 1-minimal partition of $\Omega$ is hexagonal, and therefore

$$\lim_{k\to+\infty} \frac{\mathfrak{L}_{k,1}(\Omega)|\Omega|}{k} = \lambda_1(\mathrm{H}).$$

# Minimal partitions the plane

Here we present some results and open questions stated by Bonnaillie-Noël, Helffer, and Vial (2010), and make a connection with our problems.

We define a strong, bounded and locally finite partition of $\mathbb{R}^2$ as a family of disjoint open sets $\mathcal{D} = (D_i)_{i\in\mathbb{N}}$ such that $\bigcup_{i\in\mathbb{N}} \overline{D}_i = \mathbb{R}^2$, each $D_i$ is bounded, and every disk $B(x_0, R)$ is contained in a finite union of $\overline{D}_i$'s. We say that such a partition is minimal if, for each finite subset $I$ of $\mathbb{N}$ such that

$$\Omega_I = \operatorname{Int}\left(\bigcup_{i\in I} \overline{D}_i\right)$$

is connected, $(D_i)_{i\in I}$ is a minimal $k$-partition of $\Omega_I$ (for the sum or for the maximum), with $k = \sharp I$. The following statement implies the hexagonal conjecture.

# Open questions

### Conjecture

An hexagonal tiling of $\mathbb{R}^2$ is a minimal partition for the sum, in the above sense.

### Remark

If the hexagonal conjecture is true, and if there exists a minimal partition of $\mathsf{T}(1, \sqrt{3}/2)$ whose domains are homeomorphic to disks, then the hexagonal tiling by $\mathsf{H}_4(\sqrt{3}/2)$ is minimal.

The following open question may be easier.

### Problem 3

Does there exist a minimal partition of $\mathbb{R}^2$, for the sum or for the maximum?

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Thank you for your attention!