Spectral stability under removal of small capacity sets and applications to Aharonov-Bohm operators

L. Abatangelo[a], V. Felli[b], L. Hillairet[c], C. Léna[d]

November 10, 2016

Abstract

We first establish a sharp relation between the order of vanishing of a Dirichlet eigenfunction at a point and the leading term of the asymptotic expansion of the Dirichlet eigenvalue variation, as a removed compact set concentrates at that point. Then we apply this spectral stability result to the study of the asymptotic behaviour of eigenvalues of Aharonov-Bohm operators with two colliding poles moving on an axis of symmetry of the domain.

Keywords. Asymptotics of eigenvalues, small capacity sets, Aharonov-Bohm operators.

MSC classification. Primary: 35P20; Secondary: 31C15, 35P15, 35J10.

1 Introduction and main results

The present paper is concerned with asymptotic estimates of the eigenvalue variation under either removal of small sets from the domain or operator variations due to moving poles of singular coefficients. More precisely, in the first part of the paper we will investigate the relation between the order of vanishing of a Dirichlet eigenfunction at a point and the leading term of the asymptotic expansion of the Dirichlet eigenvalue variation, as a removed compact set concentrates at that point. In the second part of the paper we will consider Aharonov-Bohm operators with two poles lying on the symmetry axis of an axially-symmetric domain and study the asymptotic behaviour of eigenvalues as the poles move coalescing into a fixed point. A spectral equivalence between this class of Aharonov-Bohm operators and the Dirichlet Laplacian will be established, once the poles' joining segment has been removed. Thus sharp expansions for the Aharonov-Bohm operators will be derived from those obtained in the first part of the paper.

1.1 Eigenvalue variation estimates under removal of small capacity sets

It is well-known that the spectrum of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$ does not change when a zero capacity compact set is removed from Ω, see e.g. [19]. In the first part of the present paper we are interested in spectral stability of the Dirichlet Laplacian and estimates of the eigenvalue variations when the domain is perturbed by removing sets of small capacity: we mean the possibility that, if $K \subset \Omega$ is a compact set, the N-th Dirichlet eigenvalue $\lambda_N(\Omega \setminus K)$ in $\Omega \setminus K$ may be close to $\lambda_N(\Omega)$ if (and only if) the capacity of K in Ω is close to zero. The seminal

[a]Dipartimento di Matematica e Applicazioni, Università di Milano–Bicocca, Via Cozzi 55, 20125 Milano, Italy, laura.abatangelo@unimib.it

[b]Dipartimento di Scienza dei Materiali, Università di Milano–Bicocca, Via Cozzi 55, 20125 Milano, Italy, veronica.felli@unimib.it

[c]MAPMO (UMR6628), Université d'Orléans, 45067 Orléans, France, luc.hillairet@univ-orleans.fr

[d]Dipartimento di Matematica *Giuseppe Peano*, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy, clena@unito.it

work [19] excited much interest and now a wide literature deals with this topic, showing that a perturbation theory can be developed in this situation.

We consider a bounded, connected open set $\Omega \subset \mathbb{R}^n$. Let $K \subset \Omega$ be a compact set. The (condenser) capacity of K in Ω is defined as

$$\mathrm{Cap}_\Omega(K) = \inf \left\{ \int_\Omega |\nabla f|^2 : \ f \in H^1_0(\Omega) \text{ and } f - \eta_K \in H^1_0(\Omega \setminus K) \right\}, \tag{1}$$

where η_K is a fixed smooth function such that $\operatorname{supp} \eta_K \subset \Omega$ and $\eta_K \equiv 1$ in a neighborhood of K. It is easy to prove that the infimum (1) is achieved by a function $V_K \in H^1_0(\Omega)$ such that $V_K - \eta_K \in H^1_0(\Omega \setminus K)$, so that

$$\mathrm{Cap}_\Omega(K) = \int_\Omega |\nabla V_K|^2 \, dx, \tag{2}$$

where V_K (capacitary potential) is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta V_K = 0, & \text{in } \Omega \setminus K, \\ V_K = 0, & \text{on } \partial\Omega, \\ V_K = 1, & \text{on } K. \end{cases} \tag{3}$$

By saying that V_K solves (3) we mean that $V_K \in H^1_0(\Omega)$, $V_K - \eta_K \in H^1_0(\Omega \setminus K)$, and

$$\int_{\Omega\setminus K} \nabla V_K \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1_0(\Omega \setminus K).$$

In [12], Courtois proves spectral stability under removal of small capacity sets in a very general context; furthermore, [12] shows that, when $K \subset \Omega$ is a compact set with $\mathrm{Cap}_\Omega(K)$ close to zero, then the function

$$\lambda_N(\Omega \setminus K) - \lambda_N(\Omega) \tag{4}$$

is even differentiable with respect to $\mathrm{Cap}_\Omega(K)$. More precisely, in [12] the following result is established.

Theorem 1.1. [12, Theorem 1.2] *Let X be a compact Riemannian manifold. Let $\lambda := \lambda_N = \ldots = \lambda_{N+k-1}$ be a Dirichlet eigenvalue of X with multiplicity k. There exist a function $r : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0} r(t) = 0$ and a positive constant ε_N, such that, for any compact set A of X, if $\mathrm{Cap}_X(A) \le \varepsilon_N$, then*

$$|\lambda_{N+j}(X \setminus A) - \lambda_{N+j} - \mathrm{Cap}_X(A) \cdot \mu_A(u^2_{N+j})| \le \mathrm{Cap}_X(A) \cdot r(\mathrm{Cap}_X(A)) \tag{5}$$

where μ_A is a finite positive probability measure supported in A defined as the renormalized singular part of ΔV_A and $\{u_N, \ldots, u_{N+k-1}\}$ is an orthonormal basis of the eigenspace of λ which diagonalises the quadratic form $\mu_A(u^2)$ according to the increasing order of its eigenvalues.

We mention that, in the particular case of A concentrating to a point (see Definition 1.2) estimate (5) is proved by Flucher in [14, Theorem 6]. Theorem 1.1 above provides a sharp asymptotic expansion of $\lambda_{N+j}(X \setminus A) - \lambda_{N+j}$ as $\mathrm{Cap}_X(A) \to 0$ if $\mu_A(u^2_{N+j}) \not\to 0$, but in general it reduces just to estimate the difference $\lambda_{N+j}(X \setminus A) - \lambda_{N+j}$ without giving its sharp vanishing order when $\mu_A(u^2_{N+j}) \to 0$. A sharp asymptotic expansion of the eigenvalue variation in the case of $\mu_A(u^2_{N+j})$ vanishing requires a more precise estimate than (5). In this regard, scanning through the proof of Theorem 1.1 given in [12] (see also [14, Theorem 7]), one realizes that when the eigenvalue $\lambda_N(\Omega)$ is simple the significant quantity is instead the u_N-capacity defined below, u_N being a (L^2-normalized) eigenfunction related to $\lambda_N(\Omega)$. Indeed, this can better describe the expansion of eigenvalues' variation, as stated in Theorem 1.4 below.

For every $u \in H^1_0(\Omega)$, we defined the u-capacity as

$$\mathrm{Cap}_\Omega(K, u) = \inf \left\{ \int_\Omega |\nabla f|^2 : \ f \in H^1_0(\Omega) \text{ and } f - u \in H^1_0(\Omega \setminus K) \right\}. \tag{6}$$

We note that when $u = 1$ in a neighborhood of K, then we recover the definition (1) of the condenser capacity. Definition (6) can be extended to $H^1_{\mathrm{loc}}(\Omega)$ functions, just defining, for any $u \in H^1_{\mathrm{loc}}(\Omega)$, $\mathrm{Cap}_\Omega(K, u) := \mathrm{Cap}_\Omega(K, \eta_K u)$ being η_K as in (1).

The infimum in (6) is achieved by a function $V_{K,u}$ which is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta V_{K,u} = 0, & \text{in } \Omega \setminus K, \\ V_{K,u} = 0, & \text{on } \partial\Omega, \\ V_{K,u} = u, & \text{on } K, \end{cases} \tag{7}$$

in such a way that

$$\mathrm{Cap}_\Omega(K, u) = \int_\Omega |\nabla V_{K,u}|^2 \, dx. \tag{8}$$

By saying that $V_{K,u}$ solves (7) we mean that $V_{K,u} \in H^1_0(\Omega)$, $V_{K,u} - u \in H^1_0(\Omega \setminus K)$, and

$$\int_{\Omega\setminus K} \nabla V_{K,u} \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in H^1_0(\Omega \setminus K). \tag{9}$$

We refer to [12, Section 2] for description of the properties of the u-capacity and to [5, Section 2] for the specific case of the u_1-capacity (which is also called *Dirichlet capacity*). For our purposes, it is important to observe the continuity properties of the f-capacity for family of concentrating compact sets described in the remark below.

Definition 1.2. *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω. We say that K_ε is* concentrating to a compact set $K \subset \Omega$ *if for every open set $U \subseteq \Omega$ such that $U \supset K$ there exists $\varepsilon_U > 0$ such that $U \supset K_\varepsilon$ for every $\varepsilon < \varepsilon_U$.*

Remark 1.3. *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω concentrating to a compact set $K \subset \Omega$ such that one of the two following conditions hold:*

(i) $\mathrm{Cap}_\Omega(K) = 0$;

(ii) $K = \bigcap_{\varepsilon>0} K_\varepsilon$ *where K_ε is decreasing as $\varepsilon \to 0$ (i.e. $K_{\varepsilon_1} \subseteq K_{\varepsilon_2}$ if $\varepsilon_1 > \varepsilon_2$).*

Then, for all $f \in H^1_0(\Omega)$, $V_{K_\varepsilon,f} \to V_{K,f}$ strongly in $H^1_0(\Omega)$ and $\lim_{\varepsilon\to0^+} \mathrm{Cap}_\Omega(K_\varepsilon, f) = \mathrm{Cap}_\Omega(K, f)$; in particular, $V_{K_\varepsilon} \to V_K$ in $H^1_0(\Omega)$ and $\lim_{\varepsilon\to0^+} \mathrm{Cap}_\Omega(K_\varepsilon) = \mathrm{Cap}_\Omega(K)$.

The proof in the case of assumption (ii) can be found in [12, Proposition 2.4]; for case (i) we refer to Proposition B.1 in the appendix.

The following result is essentially contained in the intermediate steps which are developed in [12] to prove estimate (5). It provides a sharp asymptotic expansion of (4) in terms of the u_N-capacity when the eigenvalue $\lambda_N(\Omega)$ is simple. We observe that the derivation of (5) for non simple eigenvalues requires an estimate of the remaining term in the asymptotic expansion uniformly with respect to all eigenfunctions: this is performed in [12] in terms of the condenser capacity. On the other hand, for a simple eigenvalue, the intermediate estimates obtained in [12, formulas (31) and (50)] in terms of the u_N-capacity are enough to obtain the following sharp asymptotic expansion.

Theorem 1.4. *Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian in a bounded, connected, and open set $\Omega \subset \mathbb{R}^n$ and let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to $\lambda_N(\Omega)$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact sets contained in Ω concentrating to a compact set K with* $\mathrm{Cap}_\Omega(K) = 0$. *Then*

$$\lambda_N(\Omega \setminus K_\varepsilon) = \lambda_N(\Omega) + \mathrm{Cap}_\Omega(K_\varepsilon, u_N) + o\left(\mathrm{Cap}_\Omega(K_\varepsilon, u_N)\right), \quad \textit{as } \varepsilon \to 0.$$

As already mentioned, the proof of Theorem 1.4 is contained in the proof of [12, Theorem 1.2], which is based on a method of approximation of small eigenvalues introduced in [11] (see also [12, Proposition 3.1]). Nevertheless, for the sake of clarity and completeness, we present an alternative proof in the appendix, which relies on the use of the spectral theorem to estimate the eigenvalue variation.

As observed in [12, Proposition 2.8], for every eigenfunction u of the Dirichlet Laplacian in Ω, we have that $\mathrm{Cap}_\Omega(K_\varepsilon, u) = O(\mathrm{Cap}_\Omega(K_\varepsilon))$ as $\varepsilon \to 0$. This in particular means that Theorem 1.4 is sharper than Theorem 1.1 since even the remaining term is estimated in terms of the u_N-capacity.

We mention that estimates from above and below (but not sharp asymptotic expansions) of the eigenvalue variation in terms of the u_1-capacity were obtained in [5], in the case of a compact Riemannian manifold with boundary with a small subset removed.

Motivated by Theorems 1.1 and 1.4, we devote the first part of the present paper to the derivation of the asymptotics of the key quantity $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ (which tends continuously to 0 as K_ε concentrates to a compact zero-capacity set, as observed in remark 1.3) with the goal of writing the sharp asymptotic expansion of (4) in some relevant examples. In particular we address the case of compact sets concentrating to a point, which has indeed zero capacity in any dimension greater than or equal to 2. We will show that the asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ depends on the limit point, more precisely on the order of vanishing of u_N at that point.

As a first remark in this direction, if the eigenfunction u_N does not vanish at the limit point, then the u_N-capacity is in fact asymptotic to the condenser capacity (up to a constant).

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with $n \geq 2$, let $u \in H^1_0(\Omega) \cap C^2(\Omega)$ and $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact sets contained in Ω concentrating to a point $x_0 \in \Omega$ such that $u(x_0) \neq 0$. Then*

$$\mathrm{Cap}_\Omega(K_\varepsilon, u) = u^2(x_0)\mathrm{Cap}_\Omega(K_\varepsilon) + o(\mathrm{Cap}_\Omega(K_\varepsilon)), \qquad \text{as } \varepsilon \to 0. \tag{10}$$

In view of Proposition 1.5, if the eigenfunction u does not vanish at x_0, then the u-capacity is asymptotic to the condenser capacity and the problem of sharp asymptotics of the eigenvalue variation (4) for K concentrating at x_0 is reduced to the study of the behaviour of $\mathrm{Cap}_\Omega(K)$. In dimension 2 we succeed in proving the following sharp asymptotic expansion of the condenser capacity of generic compact connected sets concentrating to a point in terms of their diameter.

Proposition 1.6. *Let Ω be a bounded connected open set $\Omega \subset \mathbb{R}^2$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact connected sets contained in Ω concentrating to a point $x_0 \in \Omega$. Let $\delta_\varepsilon = \mathrm{diam}\, K_\varepsilon$, so that $\delta_\varepsilon \to 0^+$ as $\varepsilon \to 0^+$. Then*

$$\mathrm{Cap}_\Omega(K_\varepsilon) = \frac{2\pi}{|\log(\delta_\varepsilon)|} + O\left(\frac{1}{\log^2(\delta_\varepsilon)}\right), \qquad \text{as } \varepsilon \to 0^+.$$

The proof of Proposition 1.6 is based on Steiner symmetrization methods together with elliptic coordinates, see section 2.2.

As a consequence of Theorem 1.4, Propositions 1.5 and 1.6, we deduce the following sharp asymptotic expansion of the eigenvalue variation (4) as the removed connected compact set K concentrates to a point in dimension $n = 2$.

Theorem 1.7. *Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian in a bounded, connected, open set $\Omega \subset \mathbb{R}^2$ with the $L^2(\Omega)$-normalized associated eigenfunction u_N. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact connected sets contained in Ω concentrating to a point $x_0 \in \Omega$ such that $u_N(x_0) \neq 0$. Then*

$$\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N(\Omega) = u_N^2(x_0)\frac{2\pi}{|\log \delta_\varepsilon|} + o\left(\frac{1}{|\log \delta_\varepsilon|}\right), \qquad \text{as } \varepsilon \to 0.$$

It is worthwhile mentioning that there is a rich literature dealing with the asymptotic expansion of the eigenvalues when small sets are removed from the domain, in particular when the removed set is a tubular neighborhood of a submanifold. Theorem 1.7 above has the following counterpart in [12, Theorem 1.4], which provides the asymptotic expansion for $\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N(\Omega)$ when K_ε is a tubular neighborhood of a closed submanifold Y of codimension $p \geq 2$.

Theorem 1.8. [12, Theorem 1.4] *Let $\lambda := \lambda_N = \ldots = \lambda_{N+k-1}$ be an eigenvalue of X with multiplicity k. Let $\{u_N, \ldots, u_{N+k-1}\}$ be an orthonormal basis of the eigenspace of λ which diagonalises the quadratic form $\int_Y u^2$ according to the increasing order of its eigenvalues. Then, if K_ε is a tubular neighborhood of a closed submanifold Y of codimension $p \geq 2$, we have for $j = 0, 1, \ldots, k-1$*

$$\lambda_{N+j}(X \setminus K_\varepsilon) - \lambda_{N+j} = \phi_p(\varepsilon) \int_Y u_{N+j}^2 + o(\phi_p(\varepsilon))$$

where $\phi_p(\varepsilon) = \frac{2\pi}{|\log \varepsilon|}$ if $p = 2$ and $\phi_p(\varepsilon) = (p-2)(\mathrm{Vol}(Y))^{p-1}\varepsilon^{p-2}$ if $p \geq 3$.

Theorem 1.8 generalizes preexisting results obtained for simple eigenvalues by Ozawa [18] when K is a point and Ω is a smooth bounded domain in $\mathbb{R}^2$ and by Chavel and Feldman for any codimension p [10]. Concerning the case in which K is a point, it is worthwhile citing also [6], which provides the whole asymptotic expansion for (4). We highlight that, in the case $n = 2$ and for simple limit eigenvalues, Theorem 1.7 holds for general families of compact sets concentrating at a point, which are not required to have necessarily the special form of decreasing neighborhoods of the limit point. The validity of the asymptotic expansion for general families of removed compact sets finds applications in the analysis of spectral stability for magnetic Aharonov–Bohm operators with two coalescing points; this case requires the possibility of choosing as K_ε a nodal line of a magnetic eigenfunction joining the poles, see section 3.3 and [3].

When the limit eigenfunction u_N vanishes on the limit compact set, both Theorems 1.8 and 1.7 reduces to be just an estimate of the vanishing rate of the eigenvalue variation, without giving any sharp information on the leading term of the expansion. Nevertheless, in view of Theorem 1.4, a sharp asymptotics for simple eigenvalues can be obtained once the asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ is computed, as we will do at least for special shapes of concentrating compact sets (i.e. segments and disks) in dimension 2.

Let u be an eigenfunction of the Dirichlet Laplacian in Ω, with Ω being a bounded, connected open set in $\mathbb{R}^2$ containing 0. It is well-known that $u \in C^\infty(\Omega)$ and there exist $k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, \pi)$ such that

$$r^{-k} u(r(\cos t, \sin t)) \to \beta \sin(\alpha - kt), \tag{11}$$

in $C^{1,\tau}([0, 2\pi])$ as $r \to 0^+$ for any $\tau \in (0, 1)$ (see e.g. [13]). In this case we say that u has a zero of order k at 0. We note that (11) implies that u has exactly k nodal lines dividing the 2π-angle in equal parts; the minimal slope of tangents to nodal lines is equal to $\frac{\alpha}{k}$. We also observe that, if $k = 0$ in (11), then $\beta \sin \alpha = u(0)$.

The following result provides the asymptotics of the u-capacity in the case of segments concentrating at a point.

Theorem 1.9. *Let $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. For u being a $L^2(\Omega)$-normalized eigenfunction of the Dirichlet Laplacian in an open, bounded, connected set $\Omega \subset \mathbb{R}^2$ containing 0, let $k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha \in [0, \pi)$ be as in (11).*

(i) *If $\alpha \neq 0$, then*

$$\mathrm{Cap}_\Omega(s_\varepsilon, u) = \begin{cases} \frac{2\pi}{|\log \varepsilon|}\, u^2(0)\,(1 + o(1)), & \text{if } k = 0, \\ \varepsilon^{2k}\, \pi\, \beta^2 \sin^2 \alpha\, C_k (1 + o(1)), & \text{if } k \geq 1, \end{cases} \tag{12}$$

as $\varepsilon \to 0^+$, C_k being a positive constant depending on k (see (20)).

(ii) *If $\alpha = 0$, then $\mathrm{Cap}_\Omega(s_\varepsilon, u) = O\left(\varepsilon^{2k+2}\right)$ as $\varepsilon \to 0^+$.*

Combining Theorem 1.4 with Theorem 1.9 we obtain the following result.

Theorem 1.10. *Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian in an open, bounded, connected set $\Omega \subset \mathbb{R}^2$ containing 0, with the $L^2(\Omega)$-normalized associated eigenfunction u_N. Let*

$k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha \in [0, \pi)$ be as in expansion (11) *for u_N. For $\varepsilon > 0$ small, let $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. Then*

$$\lambda_N(\Omega \setminus s_\varepsilon) - \lambda_N(\Omega) = \begin{cases} \frac{2\pi}{|\log \varepsilon|} \, u_N^2(0) \, (1 + o(1)), & \text{if } k = 0, \ \alpha \neq 0, \\ \varepsilon^{2k} \, \pi \, \beta^2 \sin^2 \alpha \, C_k (1 + o(1)), & \text{if } k \geq 1, \ \alpha \neq 0, \\ O\left(\varepsilon^{2k+2}\right), & \text{if } \alpha = 0, \end{cases}$$

as $\varepsilon \to 0^+$.

Remark 1.11. *We observe that the condition $\alpha = 0$ means the segment s_ε to be tangent to a nodal line of the limit eigenfunction u_N. Hence Theorem 1.10 provides sharp asymptotics of $\lambda_N(\Omega \setminus s_\varepsilon)$ if the segment is transversal to nodal lines of u_N, whereas it gives just an estimate on the vanishing order of $\lambda_N(\Omega \setminus s_\varepsilon) - \lambda_N(\Omega)$ when the segment is tangent to a nodal line. In this case we expect that the vanishing order will depend on the precision of the approximation between the nodal line and the segment (e.g. if the nodal line is straight, we have trivially that the* $\mathrm{Cap}_\Omega(s_\varepsilon, u)$ *is zero and $\lambda_N(\Omega \setminus s_\varepsilon) - \lambda_N(\Omega) = 0$).*

Remark 1.12. *In the case $k = 1$, i.e. if 0 is a regular point in the nodal set of u_N, we have that $\beta^2 = |\nabla u_N(0)|^2$, hence the asymptotic expansion in Theorem 1.10 has the form*

$$\lambda_N(\Omega \setminus s_\varepsilon) - \lambda_N(\Omega) = \varepsilon^2 \, \pi \, |\nabla u_N(0)|^2 \sin^2 \alpha \, C_k (1 + o(1)), \quad \text{as } \varepsilon \to 0.$$

Another relevant example in which $\mathrm{Cap}_\Omega(K_\varepsilon, u)$ can be sharply estimated in terms of the vanishing order of u is given by small disks concentrating at a zero point of u.

Theorem 1.13. *Let $B_\varepsilon = \overline{B}(0, \varepsilon) = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{x_1^2 + x_2^2} \leq \varepsilon\}$. For u being an $L^2(\Omega)$-normalized eigenfunction of the Dirichlet Laplacian in an open, bounded, connected set $\Omega \subset \mathbb{R}^2$ containing 0, let $k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, \pi)$ be as in* (11)*. Then*

$$\mathrm{Cap}_\Omega(B_\varepsilon, u) = \begin{cases} \frac{2\pi}{|\log \varepsilon|} \, u^2(0) \, (1 + o(1)), & \text{if } k = 0, \\ 2k \, \pi \, \varepsilon^{2k} \, \beta^2 (1 + o(1)), & \text{if } k \geq 1, \end{cases} \tag{13}$$

as $\varepsilon \to 0^+$.

Combining Theorem 1.4 and Theorem 1.13, we obtain the following result.

Theorem 1.14. *Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian in an open, bounded, connected set $\Omega \subset \mathbb{R}^2$ containing 0 with the $L^2(\Omega)$-normalized associated eigenfunction u_N. Let $k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha \in [0, \pi)$ be as in expansion* (11) *for u_N. Then*

$$\lambda_N(\Omega \setminus B_\varepsilon) - \lambda_N(\Omega) = \begin{cases} \frac{2\pi}{|\log \varepsilon|} \, u_N^2(0) \, (1 + o(1)), & \text{if } k = 0, \\ 2k \, \pi \, \varepsilon^{2k} \, \beta^2 \, (1 + o(1)), & \text{if } k \geq 1, \end{cases}$$

as $\varepsilon \to 0^+$.

Remark 1.15. *In the special case $k = 1$, that is to say if 0 is a regular point in the nodal set of u_N, Theorem 1.14 gives the asymptotic expansion*

$$\lambda_N(\Omega \setminus B_\varepsilon) - \lambda_N(\Omega) = 2 \, \pi \, \varepsilon^2 \, |\nabla u_N(0)|^2 (1 + o(1)), \quad \text{as } \varepsilon \to 0.$$

1.2 Aharonov–Bohm potentials with varying poles

The special attention devoted to planar domains in the first part of the paper is well understood in the context of the applications given in the second part to the problem of spectral stability for

Aharonov–Bohm potentials with varying poles. For $a = (a_1, a_2) \in \mathbb{R}^2$, the so-called Aharonov-Bohm magnetic potential with pole a and circulation $1/2$ is defined as

$$A_a(x) = \frac{1}{2}\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\}.$$

The set of papers [1, 2, 4, 8, 17] deals with the dependence on the pole a of the spectrum of Schrödinger operators with Aharonov-Bohm vector potentials, i.e. of operators $(i\nabla + A_a)^2$ acting on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

In particular, the aforementioned set of papers provides a complete picture of sharp asymptotics for simple eigenvalues when the pole a is moving in $\overline{\Omega}$. Of course, one can consider even potentials which are sum of different Aharonov–Bohm potentials with poles located at different points in the domain, being the differential Schrödinger operator defined analogously. Concerning this, in [16] the author proves continuity of eigenvalues for Schrödinger operators with different Aharonov–Bohm potentials even in the case of coalescing poles. As an application of the results proved in the first part of the present paper, in section 3 we begin to tackle the problem of coalescing poles, looking for sharp asymptotics for simple eigenvalues. In this direction, we obtain the following result under a symmetry assumption on the domain.

Theorem 1.16. *Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(x_1, x_2) = (x_1, -x_2)$. Let Ω be an open, bounded, and connected set in $\mathbb{R}^2$ satisfying $\sigma(\Omega) = \Omega$ and $0 \in \Omega$. Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian on Ω and u_N be a $L^2(\Omega)$-normalized eigenfunction associated to $\lambda_N(\Omega)$. Let $k \in \mathbb{N} \cup \{0\}$, $\beta \in \mathbb{R} \setminus \{0\}$, and $\alpha \in [0, \pi)$ be as in expansion* (11) *for u_N and assume that $\alpha \neq 0$.*

For $a > 0$ small, let $a^- = (-a, 0)$ and $a^+ = (a, 0)$ be the poles of the following Aharonov–Bohm potential

$$A_{a^-,a^+}(x) := -A_{a^-} + A_{a^+} = -\frac{1}{2}\frac{(-x_2, x_1 + a)}{(x_1 + a)^2 + x_2^2} + \frac{1}{2}\frac{(-x_2, x_1 - a)}{(x_1 - a)^2 + x_2^2}$$

and let λ_N^a be the N-th eigenvalue for $(i\nabla + A_{a^-,a^+})^2$. Then

$$\lambda_N^a - \lambda_N(\Omega) = \begin{cases} \frac{2\pi}{|\log a|}\,|u_N(0)|^2\,(1 + o(1)), & \text{if } k = 0, \\ a^{2k}\,\pi\,\beta^2 \sin^2\alpha\, C_k(1 + o(1)), & \text{if } k \geq 1, \end{cases}$$

as $a \to 0^+$, being C_k a positive constant depending only on k (see (20)*).*

We observe that the assumption $\alpha \neq 0$ means that the poles are moving along a line which is in fact not tangent to any nodal line of the limit eigenfunction u_N.

The main idea behind the proof of Theorem 1.16 is the spectral equivalence between the Aharonov–Bohm operator in an axially symmetric domain and the Dirichlet Laplacian in the domain obtained by removing either the segment joining the poles or its complement in the axis. Such isospectrality result is established in section 3.3 and extends the isospectrality result proved in [7] for a single pole to the case of two poles. Once the spectral equivalence is established, Theorem 1.16 follows as an application of Theorem 1.10.

A weakening of the symmetry assumption required in Theorem 1.16 above presents some significant additional difficulties due to the general shape of nodal lines of eigenfunctions (i.e. they are not necessarily a straight segment); this problem is treated in [3] in the case $k = 0$.

2 u-capacity

We devote this section to explicit calculations of u-capacity in several situations.

2.1 General compact sets concentrating away from zeros

In this subsection, we present the case of general domains which are concentrating to a point away from zeros of the eigenfunction u and prove the asymptotic relation, stated in Proposition 1.5, between the u-capacity and the condenser capacity. In order to derive such asymptotics we first state the following lemma, which essentially rewrites [12, formula (53)] in a form which is more convenient for our purposes.

Lemma 2.1. *Let Ω be a bounded, connected open set in $\mathbb{R}^n$ and let K be a compact set in Ω. Let η be any smooth function such that $\operatorname{supp}\eta \subset \Omega$ and $\eta \equiv 1$ in a neighborhood of K. If $u \in H^1_0(\Omega) \cap C^2(\Omega)$ then*

$$\mathrm{Cap}_\Omega(K,u) = \mathcal{L}(u,K) - \int_\Omega V_{K,u} V_K \Delta(\eta u)\,dx - 2\int_\Omega V_{K,u} \nabla V_K \cdot \nabla(\eta u)\,dx \tag{14}$$

where

$$\left(\min_{x\in K} u^2(x)\right)\mathrm{Cap}_\Omega(K) \le \mathcal{L}(u,K) \le \left(\max_{x\in K} u^2(x)\right)\mathrm{Cap}_\Omega(K). \tag{15}$$

Proof. Let us first assume that K is a regular compact set, meaning that K is the closure an open smooth set. Then

$$\begin{aligned}
\int_\Omega |\nabla V_{K,u}|^2\,dx &= \int_{\Omega\setminus K} |\nabla V_{K,u}|^2\,dx + \int_K |\nabla u|^2\,dx \\
&= \int_{\partial(\Omega\setminus K)} V_{K,u}\partial_\nu V_{K,u}\,d\sigma + \int_{\partial K} u\partial_\nu u\,d\sigma - \int_K u\Delta u\,dx \\
&= \int_{\partial(\Omega\setminus K)} V_K u\partial_\nu V_{K,u}\,d\sigma + \int_{\partial K} u\partial_\nu u\,d\sigma - \int_K u\Delta(\eta u)\,dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{\partial(\Omega\setminus K)} V_K u\partial_\nu V_{K,u}\,d\sigma = \int_{\partial(\Omega\setminus K)} V_K(\eta u)\partial_\nu V_{K,u}\,d\sigma = \int_{\Omega\setminus K} \nabla\left(V_K\eta u\right)\cdot\nabla V_{K,u}\,dx \\
&= \int_{\Omega\setminus K} \eta u\nabla V_K\cdot\nabla V_{K,u}\,dx + \int_{\Omega\setminus K} V_K\nabla(\eta u)\cdot\nabla V_{K,u}\,dx \\
&= \int_{\partial(\Omega\setminus K)} uV_{K,u}\partial_\nu V_K\,d\sigma - \int_{\Omega\setminus K} V_{K,u}\nabla V_K\cdot\nabla(\eta u)\,dx + \int_{\partial(\Omega\setminus K)} V_K V_{K,u}\partial_\nu(\eta u)\,d\sigma \\
&\quad - \int_{\Omega\setminus K} V_K V_{K,u}\Delta(\eta u)\,dx - \int_{\Omega\setminus K} V_{K,u}\nabla V_K\cdot\nabla(\eta u)\,dx \\
&= \int_{\partial(\Omega\setminus K)} u^2\partial_\nu V_K\,d\sigma - \int_{\partial K} u\partial_\nu u\,d\sigma - \int_{\Omega\setminus K} V_K V_{K,u}\Delta(\eta u)\,dx - 2\int_{\Omega\setminus K} V_{K,u}\nabla V_K\cdot\nabla(\eta u)\,dx.
\end{aligned}$$

Hence we obtain that, if K is regular, then

$$\mathrm{Cap}_\Omega(K,u) = \int_{\partial(\Omega\setminus K)} u^2\partial_\nu V_K\,d\sigma - \int_\Omega V_K V_{K,u}\Delta(\eta u)\,dx - 2\int_\Omega V_{K,u}\nabla V_K\cdot\nabla(\eta u)\,dx. \tag{16}$$

If K is a generic compact set, then there exist a decreasing family of regular compact sets $\{K_\varepsilon\}_{\varepsilon>0}$ concentrating at K such that $K = \bigcap_{\varepsilon>0} K_\varepsilon$. If $\eta \in C^\infty_c(\Omega)$ is any smooth function such that $\eta \equiv 1$ in a neighborhood of K, then $\eta \equiv 1$ also in a neighborhood of K_ε for ε sufficiently small. Writing (16) for K_ε and η and passing to the limit, in view of Remark 1.3 (case (ii)) we obtain that

$$\mathrm{Cap}_\Omega(K,u) = \mathcal{L}(u,K) - \int_\Omega V_K V_{K,u}\Delta(\eta u)\,dx - 2\int_\Omega V_{K,u}\nabla V_K\cdot\nabla(\eta u)\,dx$$

with $\mathcal{L}(u,K) = \lim_{\varepsilon\to 0^+} \int_{\partial(\Omega\setminus K_\varepsilon)} u^2 \partial_\nu V_{K_\varepsilon}\, d\sigma$. By Hopf's Lemma, $\partial_\nu V_{K_\varepsilon}$ is positive on ∂K_ε, being ν the exterior normal vector to $\Omega\setminus K_\varepsilon$. Moreover, by integration by parts, we have that $\int_{\partial K_\varepsilon} |\partial_\nu V_{K_\varepsilon}|\, d\sigma = \mathrm{Cap}_\Omega(K_\varepsilon)$. Hence

$$\left(\min_{K_\varepsilon} u^2\right)\mathrm{Cap}_\Omega(K_\varepsilon) \le \min_{\partial K_\varepsilon} u^2 \int_{\partial K_\varepsilon} |\partial_\nu V_{K_\varepsilon}|\, d\sigma$$
$$\le \int_{\partial(\Omega\setminus K_\varepsilon)} u^2 \partial_\nu V_{K_\varepsilon}\, d\sigma \le \max_{\partial K_\varepsilon} u^2 \int_{\partial K_\varepsilon} |\partial_\nu V_{K_\varepsilon}|\, d\sigma \le \left(\max_{K_\varepsilon} u^2\right)\mathrm{Cap}_\Omega(K_\varepsilon).$$

By Remark 1.3 (case (ii)) and continuity of u, passing to the limit in the above estimate yields (15), thus completing the proof. □

From Lemma 2.1 we derive Proposition 1.5.

Proof of Proposition 1.5. Let $\eta \in C^\infty_c(\Omega)$ be a smooth function such that $\eta \equiv 1$ in a neighborhood of x_0, so that (14) can be written for K_ε and η for ε sufficiently small. The fact that K_ε concentrates to x_0 as $\varepsilon \to 0$ and the continuity of u implies that

$$\lim_{\varepsilon\to 0} \min_{K_\varepsilon} u^2 = \lim_{\varepsilon\to 0} \max_{K_\varepsilon} u^2 = u^2(x_0),$$

so that $\mathcal{L}(u, K_\varepsilon) = u^2(x_0)\mathrm{Cap}_\Omega(K_\varepsilon) + o(\mathrm{Cap}_\Omega(K_\varepsilon))$ as $\varepsilon \to 0^+$. From Cauchy-Schwarz Inequality and Corollary A.2 we deduce that

$$\left|\int_\Omega V_{K_\varepsilon} V_{K_\varepsilon,u} \Delta(\eta u)\, dx\right| \le \|\Delta(\eta u)\|_{L^\infty(\Omega)} \|V_{K_\varepsilon}\|_{L^2(\Omega)} \|V_{K_\varepsilon,u}\|_{L^2(\Omega)} = o\left(\mathrm{Cap}_\Omega(K_\varepsilon)\right),$$

as $\varepsilon \to 0^+$. According to Cauchy-Schwarz Inequality and Corollary A.2,

$$\left|\int_\Omega V_{K_\varepsilon,u_N} \nabla V_{K_\varepsilon}\cdot\nabla(\eta u)\, dx\right| \le \|\nabla(\eta u)\|_{L^\infty(\Omega)} \|\nabla V_{K_\varepsilon}\|_{L^2(\Omega)} \|V_{K_\varepsilon,u_N}\|_{L^2(\Omega)} = o\left(\mathrm{Cap}_\Omega(K_\varepsilon)\right),$$

as $\varepsilon \to 0^+$. Equation (10) then follows from (14). □

2.2 Capacities in dimension 2

In this subsection we present some explicit computations for capacities of compact sets concentrating to a point in a planar domain.

To this aim, we first derive the following estimate of the h-capacity in terms of the vanishing order of the function h at the concentration point of compact sets.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with $n \ge 2$ and $0 \in \Omega$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω such that, for some $C > 0$ and ε sufficiently small,*

$$K_\varepsilon \subset \overline{B}(0, C\varepsilon).$$

Let $h \in H^1(\Omega)$ be such that $h(x) = O(|x|^{k+1})$ and $|\nabla h(x)| = O(|x|^k)$ as $|x| \to 0$ for some $k \in \mathbb{N}\cup\{0\}$. Then

$$\mathrm{Cap}_\Omega(K_\varepsilon, h) = O(\varepsilon^{2k+n}) \quad \text{as } \varepsilon \to 0.$$

Proof. By monotonicity of the h-capacity, it is enough to prove that

$$\mathrm{Cap}_\Omega\left(\overline{B}(0, C\varepsilon), h\right) = O\left(\varepsilon^{2k+n}\right) \text{ as } \varepsilon \to 0.$$

To do this, let us fix a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ supported in $B(0,2)$ and equal to 1 on $\overline{B}(0,1)$. Let us define

$$\varphi_\varepsilon(x) := \varphi\left(\frac{x}{C\varepsilon}\right) \quad \text{and} \quad h_\varepsilon := \varphi_\varepsilon h.$$

The function h_ε coincides with h on $\overline{B}(0, C\varepsilon)$, so we have, by definition of the capacity,

$$\mathrm{Cap}_\Omega\left(\overline{B}(0, C\varepsilon), h\right) \leq \int_\Omega |\nabla h_\varepsilon|^2\, dx.$$

On the other hand, for any $x \in \Omega$,

$$\begin{aligned}|\nabla h_\varepsilon(x)|^2 &\leq 2\left(\varphi_\varepsilon^2(x)\, |\nabla h(x)|^2 + h^2(x)\, |\nabla \varphi_\varepsilon(x)|^2\right)\\ &= 2\left(\varphi^2\left(\frac{x}{C\varepsilon}\right) |\nabla h(x)|^2 + \frac{1}{C^2\varepsilon^2} h^2(x) \left|\nabla\varphi\left(\frac{x}{C\varepsilon}\right)\right|^2\right).\end{aligned}$$

Since $|\nabla h| = O(|x|^k)$ as $|x| \to 0$ and h_ε is supported in $B(0, 2C\varepsilon)$, then $\|\nabla h_\varepsilon\|_{L^\infty(\Omega)} \leq \mathrm{const}\, \varepsilon^k$. Therefore

$$\int_\Omega |\nabla h_\varepsilon|^2\, dx = O(\varepsilon^{2k+n}),$$

which proves the claim. □

In order to derive sharp asymptotics in both cases of condenser capacities of generic compact sets and of u-capacities of segments, a key tool is the following computation of capacity of segments in ellipses. For $L > 0$ and $\varepsilon > 0$, we denote as

$$\mathcal{E}_\varepsilon(L) = \left\{(x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{L^2 + \varepsilon^2} + \frac{x_2^2}{L^2} < 1\right\} \tag{17}$$

the interior of the ellipse centered at 0 with major semi-axis of length $\sqrt{L^2 + \varepsilon^2}$ and minor semi-axis of length L. Furthermore, for every $\varepsilon > 0$ we denote as

$$s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\} \tag{18}$$

the segment of length 2ε and center 0 on the x_1-axis.

Lemma 2.3. *Let $k \in \mathbb{N} \cup \{0\}$ and let P_k be a homogeneous polynomial of degree $k \geq 0$, i.e*

$$P_k(x_1, x_2) = \sum_{j=0}^{k} c_j {x_1}^{k-j} {x_2}^j \tag{19}$$

for some $c_0, c_1, \dots, c_k \in \mathbb{R}$. Then, for every $L > 0$,

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon(L)}(s_\varepsilon, P_k) = \begin{cases} \frac{2\pi}{|\log\varepsilon|}\, c_0^2 \left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right), & \text{if } k = 0,\\ \varepsilon^{2k}\, c_0^2\, \pi C_k (1 + o(1)), & \text{if } k \geq 1,\end{cases}$$

as $\varepsilon \to 0^+$, where

$$C_k = \sum_{j=1}^{k} j|A_{j,k}|^2, \quad \text{being} \quad A_{j,k} = \frac{1}{\pi}\int_0^{2\pi} (\cos\eta)^k \cos(j\eta)\, d\eta. \tag{20}$$

Remark 2.4. *We notice that, if $k \geq 1$, then there exists at least a $j \in \{1, 2, \dots, k\}$ such that $A_{j,k} \neq 0$, so that $C_k = \sum_{j=1}^k j|A_{j,k}|^2 \neq 0$ if $k \geq 1$.*

Remark 2.5. *As a particular case of Lemma 2.3 when $k = 0$ and $c_0 = 1$ (so that $P_k \equiv 1$), we obtain that the condenser capacity of the segment in the ellipse is given*

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon(L)}(s_\varepsilon) = \frac{2\pi}{|\log\varepsilon|}\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right) \quad \text{as } \varepsilon \to 0^+. \tag{21}$$

Proof of Lemma 2.3. We define the elliptic coordinates (ξ, η) (see for instance [21]) by

$$\begin{cases} x_1 = \varepsilon \cosh(\xi)\cos(\eta), \\ x_2 = \varepsilon \sinh(\xi)\sin(\eta), \end{cases} \qquad \xi \geq 0, \ 0 \leq \eta < 2\pi.$$

Let us note that, in these coordinates, the segment s_ε is defined by $\xi = 0$, whereas $\mathcal{E}_\varepsilon$ is defined by $0 \leq \xi < \xi_\varepsilon$ and $\partial\mathcal{E}_\varepsilon$ is described by the condition $\xi = \xi_\varepsilon$, with $\varepsilon \sinh(\xi_\varepsilon) = L$, that is to say

$$\xi_\varepsilon = \operatorname{argsinh}\left(\frac{L}{\varepsilon}\right) = \log\left(\frac{L}{\varepsilon} + \sqrt{1 + \frac{L^2}{\varepsilon^2}}\right). \tag{22}$$

A direct computation shows that the mapping $\Phi : (\xi, \eta) \mapsto (x_1, x_2)$ has a Jacobian matrix of the form

$$J(\Phi)(\xi, \eta) = h(\xi, \eta) O(\xi, \eta),$$

with $O(\xi, \eta)$ an orthonormal matrix and $h(\xi, \eta) > 0$ in $\mathbb{R}^2 \setminus s_\varepsilon$ satisfying

$$h^2(\xi, \eta) = \varepsilon^2(\cosh^2\xi - \cos^2\eta).$$

If we evaluate the homogeneous polynomial P_k in the new set of coordinates on the segment $s_\varepsilon = \{(\xi, \eta) : \ \xi = 0\}$, we end up with $Q_k(\xi, \eta) = P_k(\Phi(0, \eta)) = c_0 \varepsilon^k (\cos\eta)^k$. Let W be the Dirichlet potential of P_k in elliptic coordinates, that is

$$\begin{cases} -\Delta W = 0, & \text{in } (0, \xi_\varepsilon) \times (0, 2\pi), \\ W = 0, & \text{on } \xi = \xi_\varepsilon, \\ W = c_0\, \varepsilon^k (\cos\eta)^k, & \text{on } \xi = 0, \\ W(\xi, 0) = W(\xi, 2\pi), & \text{for all } \xi \in (0, \xi_\varepsilon). \end{cases}$$

Let us consider the Fourier expansion of W in elliptic coordinates:

$$\frac{1}{\varepsilon^k} W(\xi, \eta) = \frac{a_0(\xi)}{2} + \sum_{j \geq 1} \left(a_j(\xi)\cos(j\eta) + b_j(\xi)\sin(j\eta)\right)$$

where

$$a_j(\xi) = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{\varepsilon^k} W(\xi, \eta)\cos(j\eta)\, d\eta, \quad b_j(\xi) = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{\varepsilon^k} W(\xi, \eta)\sin(j\eta)\, d\eta,$$

from which we see that $b_j(0) = 0$ for any j and $a_j(0) = 0$ for all $j > k$. Therefore we have

$$0 = -\Delta_{(\xi,\eta)} W = \varepsilon^k \frac{a_0''(\xi)}{2} + \varepsilon^k \sum_{j \geq 1} \left((a_j''(\xi) - j^2 a_j(\xi))\cos(j\eta) + (b_j''(\xi) - j^2 b_j(\xi))\sin(j\eta)\right)$$

and imposing the boundary conditions for $\xi \in (0, \xi_\varepsilon)$ we obtain

$$a_0(\xi) = a_0(0)\left(1 - \frac{\xi}{\xi_\varepsilon}\right), \tag{23}$$

$$a_j(\xi) = a_j(0)\left(\frac{e^{j\xi}}{1 - e^{2j\xi_\varepsilon}} + \frac{e^{-j\xi}}{1 - e^{-2j\xi_\varepsilon}}\right), \qquad \text{for } j \geq 1 \tag{24}$$

$$b_j(\xi) = 0 \qquad \text{for } j \geq 0. \tag{25}$$

In this way

$$\frac{1}{\varepsilon^k} W(\xi, \eta) = \frac{a_0(\xi)}{2} + \sum_{j=1}^{k} a_j(\xi)\cos(j\eta)$$

and then by Parseval's identity

$$\iint_{(0,\xi_\varepsilon)\times(0,2\pi)} |\nabla W|^2 = \varepsilon^{2k}\frac{\pi}{2}\int_0^{\xi_\varepsilon} |a_0'(\xi)|^2\,d\xi + \varepsilon^{2k}\pi\sum_{j=1}^{k}\int_0^{\xi_\varepsilon} (|a_j'(\xi)|^2 + j^2|a_j(\xi)|^2)\,d\xi. \tag{26}$$

Let us now compute every term of the latter expression. First we have

$$\int_0^{\xi_\varepsilon} |a_0'(\xi)|^2\,d\xi = \frac{1}{\xi_\varepsilon}|a_0(0)|^2. \tag{27}$$

Secondly, for $j \geq 1$ we have

$$\begin{aligned}
\int_0^{\xi_\varepsilon} |a_j'(\xi)|^2\,d\xi &= j^2|a_j(0)|^2\int_0^{\xi_\varepsilon}\left(\frac{e^{j\xi}}{1-e^{2j\xi_\varepsilon}} - \frac{e^{-j\xi}}{1-e^{-2j\xi_\varepsilon}}\right)^2 d\xi\\
&= \frac{j}{2}|a_j(0)|^2\left(\frac{-1}{1-e^{2j\xi_\varepsilon}} + \frac{1}{1-e^{-2j\xi_\varepsilon}} - 4j\frac{\xi_\varepsilon}{(1-e^{2j\xi_\varepsilon})(1-e^{-2j\xi_\varepsilon})}\right)\\
&= \frac{j}{2}|a_j(0)|^2\left(\frac{e^{-2j\xi_\varepsilon} - e^{2j\xi_\varepsilon} - 4j\xi_\varepsilon}{(1-e^{2j\xi_\varepsilon})(1-e^{-2j\xi_\varepsilon})}\right)\\
&= \frac{j}{2}|a_j(0)|^2(1+o(1)) \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{28}$$

Finally we have

$$\begin{aligned}
\int_0^{\xi_\varepsilon} |a_j(\xi)|^2\,d\xi &= |a_j(0)|^2\int_0^{\xi_\varepsilon}\left(\frac{e^{j\xi}}{1-e^{2j\xi_\varepsilon}} + \frac{e^{-j\xi}}{1-e^{-2j\xi_\varepsilon}}\right)^2 d\xi\\
&= |a_j(0)|^2\frac{1}{2j}\left(\frac{-1}{1-e^{2j\xi_\varepsilon}} + \frac{1}{1-e^{-2j\xi_\varepsilon}} + 4j\frac{\xi_\varepsilon}{(1-e^{2j\xi_\varepsilon})(1-e^{-2j\xi_\varepsilon})}\right)\\
&= |a_j(0)|^2\frac{1}{2j}(1+o(1)) \qquad \text{as } \varepsilon\to 0.
\end{aligned} \tag{29}$$

Plugging (27), (28) and (29) into (26) we obtain

$$\iint_{(0,\xi_\varepsilon)\times(0,2\pi)} |\nabla W|^2 = \varepsilon^{2k}\frac{\pi}{2}\frac{1}{\xi_\varepsilon}|a_0(0)|^2 + \varepsilon^{2k}\pi\sum_{j=1}^{k} j|a_j(0)|^2(1+o(1)) \qquad \text{as } \varepsilon\to 0. \tag{30}$$

We note that for $k=0$ there holds $a_0(0) = 2c_0$, whereas $a_j(0) = 0$ for $j \geq 1$. Moreover, a simple calculation shows

$$\frac{1}{\xi_\varepsilon} = \frac{1}{|\log\varepsilon|} + O\left(\frac{1}{|\log\varepsilon|^2}\right)$$

as $\varepsilon \to 0^+$. On the other hand, if $k \geq 1$, then there exists at least a $j \in \{1, 2, \dots, k\}$ such that $a_j(0) \neq 0$.

We then conclude that

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon(L)}(s_\varepsilon, P_k) = \begin{cases} \frac{2\pi}{|\log\varepsilon|}c_0^2\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right), & \text{if } k = 0,\\ \varepsilon^{2k}\,\pi\left(\sum_{j=1}^k j|a_j(0)|^2\right)(1+o(1)), & \text{if } k \geq 1,\end{cases}$$

thus completing the proof. □

2.2.1 Condenser capacity in dimension 2

We first consider generic compact connected sets and prove the sharp asymptotic expansion of the condenser capacity in terms of their diameter, as stated in Proposition 1.6.

Proof of Proposition 1.6. Let $a_\varepsilon, b_\varepsilon \in K_\varepsilon$ be such that $|b_\varepsilon - a_\varepsilon| = \delta_\varepsilon$. We denote by m_ε the middle point of a_ε and b_ε, i.e. $m_\varepsilon = \frac{1}{2}(a_\varepsilon + b_\varepsilon)$. Note that $m_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Let us first derive an upper bound for $\mathrm{Cap}_\Omega(K_\varepsilon)$. There exists $R > 0$ such that $B(m_\varepsilon, R) \subset \Omega$ and $B(x_0, R) \subset \Omega$ for ε sufficiently small. According to the monotonicity properties of the capacity, we have

$$\mathrm{Cap}_\Omega(K_\varepsilon) \leq \mathrm{Cap}_{B(m_\varepsilon,R)}\overline{B}(m_\varepsilon, \delta_\varepsilon) = \mathrm{Cap}_{B(0,R)}\overline{B}(0, \delta_\varepsilon).$$

It is easy to compute $\mathrm{Cap}_{B(0,R)}\overline{B}(0, \delta_\varepsilon)$. Indeed, the radial function V defined as $V(x) = f(|x|)$ with

$$f(r) = \begin{cases} 1, & \text{if } r \leq \delta_\varepsilon, \\ \frac{\log(r/R)}{\log(\delta_\varepsilon/R)}, & \text{if } \delta_\varepsilon < r \leq R, \end{cases}$$

belongs to $H^1_0(B(0,R))$, is harmonic in $B(0,R) \setminus B(0,\delta_\varepsilon)$ and equal to 1 on $B(0,\delta_\varepsilon)$. Hence V is a capacitary potential and

$$\mathrm{Cap}_{B(0,R)}\overline{B}(0, \delta_\varepsilon) = \int_{B(0,R)} |\nabla V|^2 \, dx = 2\pi \int_{\delta_\varepsilon}^{R} \frac{dr}{r \log^2(\delta_\varepsilon/R)} = \frac{2\pi}{\log(R/\delta_\varepsilon)}.$$

We therefore have

$$\mathrm{Cap}_\Omega(K_\varepsilon) \leq \frac{2\pi}{\log(R/\delta_\varepsilon)}. \tag{31}$$

To find a lower bound for $\mathrm{Cap}_\Omega(K_\varepsilon)$ is a more delicate issue. Since Ω is bounded, there exists a length L such that $\Omega \subset \widetilde{\mathcal{E}}_\varepsilon$, where $\widetilde{\mathcal{E}}_\varepsilon$ is the interior of the ellipse centered at m_ε, whose major semi-axis has length $\sqrt{L^2 + \frac{1}{4}\delta_\varepsilon^2}$ and belongs to the straight line $\mathcal{D}_\varepsilon$ passing through a_ε and b_ε, and whose minor semi-axis has length L. By monotonicity of the capacity,

$$\mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(K_\varepsilon) \leq \mathrm{Cap}_\Omega(K_\varepsilon).$$

We now claim that, if $\widetilde{s}_\varepsilon$ denotes the segment of extremities a_ε and b_ε,

$$\mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{s}_\varepsilon) \leq \mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(K_\varepsilon). \tag{32}$$

To prove claim (32), we first consider a regular connected compact set $\widetilde{K}_\varepsilon$ such that $K_\varepsilon \subseteq \widetilde{K}_\varepsilon \subset \widetilde{\mathcal{E}}_\varepsilon$. Since $\widetilde{K}_\varepsilon$ is regular, we have that its capacitary potential $V_{\widetilde{K}_\varepsilon}$ is continuous in $\widetilde{\mathcal{E}}_\varepsilon$. For every $x \in \mathbb{R}^2$, let us denote as S_x the straight line perpendicular to $\mathcal{D}_\varepsilon$ passing through x. Let us consider the Steiner symmetrization of $V_{\widetilde{K}_\varepsilon}$ with respect to the line $\mathcal{D}_\varepsilon$ (see e.g. [9]), i.e.

$$V^*_{\widetilde{K}_\varepsilon}(x) = \inf\left\{t > 0 : \mathcal{H}^1(\{y \in S_x : V_{\widetilde{K}_\varepsilon}(y) > t\}) \leq 2\,\mathrm{dist}(x, \mathcal{D}_\varepsilon)\right\},$$

where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure.

Since $\widetilde{K}_\varepsilon$ is connected and $a_\varepsilon, b_\varepsilon \in \widetilde{K}_\varepsilon$, we have that $S_x \cap \widetilde{K}_\varepsilon \neq \emptyset$ for every $x \in \widetilde{s}_\varepsilon$. It follows that, for every $x \in \widetilde{s}_\varepsilon$, $\sup_{S_x \cap \widetilde{\mathcal{E}}_\varepsilon} V_{\widetilde{K}_\varepsilon} = 1$; then $\mathcal{H}^1(\{y \in S_x : V_{\widetilde{K}_\varepsilon}(y) > t\}) = 0$ if and only if $t \geq 1$. It follows that $V^*_{\widetilde{K}_\varepsilon}(x) = 1$ for every $x \in \widetilde{s}_\varepsilon$. Then

$$\mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{s}_\varepsilon) \leq \int_{\widetilde{\mathcal{E}}_\varepsilon} |\nabla V^*_{\widetilde{K}_\varepsilon}|^2 \, dx.$$

Since Steiner symmetrization decreases the Dirichlet energy, we obtain also that

$$\int_{\widetilde{\mathcal{E}}_\varepsilon} |\nabla V^*_{\widetilde{K}_\varepsilon}|^2 \, dx \leq \int_{\widetilde{\mathcal{E}}_\varepsilon} |\nabla V_{\widetilde{K}_\varepsilon}|^2 \, dx = \mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{K}_\varepsilon)$$

thus concluding that $\mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{s}_\varepsilon) \leq \mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{K}_\varepsilon)$. Finally, to obtain (32) it is enough to approximate K_ε by regular connected compact sets and invoke Remark 1.3 (ii).

Since a roto-translation transforms $\widetilde{\mathcal{E}}_\varepsilon$ into $\mathcal{E}_{\delta_\varepsilon/2}$ and $\widetilde{s}_\varepsilon$ into $s_{\delta_\varepsilon/2}$ (see the notations introduced in (17) and (18)), from (21) it follows that

$$\mathrm{Cap}_{\widetilde{\mathcal{E}}_\varepsilon}(\widetilde{s}_\varepsilon) = \mathrm{Cap}_{\mathcal{E}_{\delta_\varepsilon/2}}(s_{\delta_\varepsilon/2}) = \frac{2\pi}{|\log \delta_\varepsilon|}\left(1 + O\left(\frac{1}{|\log \delta_\varepsilon|}\right)\right) \quad \text{as } \varepsilon \to 0^+.$$

Putting the above inequalities and computations together, we get

$$\mathrm{Cap}_\Omega(K_\varepsilon) \geq \frac{2\pi}{|\log \delta_\varepsilon|}\left(1 + O\left(\frac{1}{|\log \delta_\varepsilon|}\right)\right) \quad \text{as } \varepsilon \to 0^+. \tag{33}$$

Putting together Equations (31) and (33), we obtain

$$\frac{2\pi}{|\log \delta_\varepsilon|}\left(1 + O\left(\frac{1}{|\log \delta_\varepsilon|}\right)\right) \leq \mathrm{Cap}_\Omega(K_\varepsilon) \leq \frac{2\pi}{\log(R/\delta_\varepsilon)}. \tag{34}$$

Observing that

$$\frac{2\pi}{\log(R/\delta_\varepsilon)} = \frac{2\pi}{|\log \delta_\varepsilon|}\left(1 + O\left(\tfrac{1}{|\log \delta_\varepsilon|}\right)\right)$$

as $\delta_\varepsilon \to 0^+$, we conclude the proof. □

2.2.2 u-capacity for segments concentrating to a point in dimension 2

We now compute the u-capacity for the special shape of segments of length 2ε centered at 0; for this particular shape we are able to consider even the case when the limit point is a zero of the eigenfunction u. The interest in computing the u-capacity of segments with coalescing extremities is motivated by the remarkable application to eigenvalue asymptotics for Aharonov-Bohm operators with two poles presented in section 3.

The following proposition gives the asymptotics for h-capacity of concentrating segments in a planar domain when h is a homogeneous polynomial.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with $0 \in \Omega$. For $\varepsilon > 0$ small, let s_ε be as in* (18) *and P_k be a homogeneous polynomial of degree $k \geq 0$ as in* (19) *for some $c_0, c_1, \dots, c_k \in \mathbb{R}$. Then*

$$\mathrm{Cap}_\Omega(s_\varepsilon, P_k) = \begin{cases} \frac{2\pi}{|\log \varepsilon|}\, c_0^2 \left(1 + O\left(\frac{1}{|\log \varepsilon|}\right)\right), & \text{if } k = 0,\\ \varepsilon^{2k}\, c_0^2\, \pi C_k(1 + o(1)), & \text{if } k \geq 1, \end{cases} \tag{35}$$

as $\varepsilon \to 0^+$, with C_k as in (20).

Proof. Since Ω is open and bounded, there exist $L_2 > L_1 > 0$ such that, for ε sufficiently small, $s_\varepsilon \subset \mathcal{E}_\varepsilon(L_1) \subset \Omega \subset \mathcal{E}_\varepsilon(L_2)$, with $\mathcal{E}_\varepsilon(L_1), \mathcal{E}_\varepsilon(L_2)$ being as in (17). By monotonicity of the capacity,

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon(L_2)}(s_\varepsilon, P_k) \leq \mathrm{Cap}_\Omega(s_\varepsilon, P_k) \leq \mathrm{Cap}_{\mathcal{E}_\varepsilon(L_1)}(s_\varepsilon, P_k).$$

The conclusion then follows from Lemma 2.3. □

The following proposition gives, for every sufficiently smooth function u, a sharp relation between the asymptotics of $\mathrm{Cap}_\Omega(s_\varepsilon, u)$ and the order of vanishing of u at $0 \in \Omega$.

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with $0 \in \Omega$ and let $k \in \mathbb{N} \cup \{0\}$. Let us assume that $u \in C^{k+1}_{\mathrm{loc}}(\Omega) \setminus \{0\}$ has vanishing order at 0 equal to k, i.e. the Taylor polynomial of u of order k and center 0 has degree k and is non-zero and k-homogeneous, namely is of the form*

$$P_k(x_1, x_2) = \sum_{j=0}^{k} c_j {x_1}^{k-j} {x_2}^j$$

for some $c_0, c_1, \dots, c_k \in \mathbb{R}$, $(c_0, c_1, \dots, c_k) \neq (0, 0, \dots, 0)$.

(i) *If* $c_0 \neq 0$, *then*

$$\mathrm{Cap}_\Omega(s_\varepsilon, u) = \begin{cases} \frac{2\pi}{|\log \varepsilon|}\, u^2(0)\big(1+o(1)\big), & \text{if } k=0,\\ \varepsilon^{2k}\, c_0^2\, \pi C_k(1+o(1)), & \text{if } k \geq 1, \end{cases} \tag{36}$$

as $\varepsilon \to 0^+$, C_k *being defined in* (20).

(ii) *If* $c_0 = 0$, *then* $\mathrm{Cap}_\Omega(s_\varepsilon, u) = O\left(\varepsilon^{2k+2}\right)$ *as* $\varepsilon \to 0^+$.

Proof. From the Taylor formula, we can write u as $u = P_k + h$ for some $h \in C^{k+1}_{\mathrm{loc}}(\Omega)$ satisfying

$$h(x) = O(|x|^{k+1}) \quad \text{and} \quad |\nabla h(x)| = O(|x|^k) \quad \text{as } |x| \to 0^+.$$

We denote by V, W_0, and W the capacitary potentials associated with the capacities $\mathrm{Cap}_\Omega(s_\varepsilon, u)$, $\mathrm{Cap}_\Omega(s_\varepsilon, P_k)$, and $\mathrm{Cap}_\Omega(s_\varepsilon, h)$ respectively. By linearity of the Dirichlet problem, $V = W_0 + W$. Therefore we have that

$$\mathrm{Cap}_\Omega(s_\varepsilon, u) = \int_\Omega |\nabla V|^2\, dx = \int_\Omega |\nabla W_0|^2\, dx + 2\int_\Omega \nabla W_0 \cdot \nabla W\, dx + \int_\Omega |\nabla W|^2\, dx.$$

By Lemma 2.2 we have that, as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla W|^2\, dx = O\left(\varepsilon^{2k+2}\right)$$

and

$$\left|\int_\Omega \nabla W_0 \cdot \nabla W\, dx\right| \leq \|\nabla W_0\|_{L^2(\Omega)} \|\nabla W\|_{L^2(\Omega)} = \sqrt{\mathrm{Cap}_\Omega(s_\varepsilon, P_k)}\, O\left(\varepsilon^{k+1}\right).$$

Hence

$$\mathrm{Cap}_\Omega(s_\varepsilon, u) = \mathrm{Cap}_\Omega(s_\varepsilon, P_k) + \sqrt{\mathrm{Cap}_\Omega(s_\varepsilon, P_k)}\, O\left(\varepsilon^{k+1}\right) + O\left(\varepsilon^{2k+2}\right), \tag{37}$$

as $\varepsilon \to 0^+$. In view of Proposition 2.6 and (37), we have that, if $c_0 \neq 0$,

$$\mathrm{Cap}_\Omega(s_\varepsilon, u) = \mathrm{Cap}_\Omega(s_\varepsilon, P_k)\,(1 + o(1))$$

as $\varepsilon \to 0^+$, from which estimate (36) follows thanks to Proposition 2.6.

On the other hand, if $c_0 = 0$, then Proposition 2.6 implies that $\mathrm{Cap}_\Omega(s_\varepsilon, P_k) = 0$, hence from (37) it follows that $\mathrm{Cap}_\Omega(s_\varepsilon, u) = O\left(\varepsilon^{2k+2}\right)$ as $\varepsilon \to 0^+$. □

The proof of Theorem 1.9 (and consequently of Theorem 1.10) now follows as a particular case of Proposition 2.7.

Proof of Theorems 1.9 and 1.10. From the fact that $u \in C^\infty(\Omega)$ and (11) it follows that the Taylor polynomial of the function u with center 0 and order k is harmonic, k-homogeneous, and has degree k; more precisely it has the form

$$P_k(r\cos t, r\sin t) = \beta r^k \sin(\alpha - kt).$$

Since it can be also written as $P_k(x_1, x_2) = \sum_{j=0}^{k} c_j {x_1}^{k-j} {x_2}^j$ for some $c_0, c_1, \dots, c_k \in \mathbb{R}$, we have then that necessarily $c_0 = \beta \sin\alpha$. The proof of Theorem 1.9 then follows from Proposition 2.7. Finally, the proof of Theorem 1.10 is a direct consequence of Theorems 1.4 and 1.9. □

2.2.3 u-capacity for small disks concentrating to a point in dimension 2

We conclude this section with a proof of Theorem 1.13. As in the proof of Theorem 1.9, we rely on explicit computation of the u-capacity in a special case. The following result is the counterpart, in the case of the disks, of Lemma 2.3, which was stated for segments.

Lemma 2.8. *Let $k \in \mathbb{N}$, $k \geq 1$, and let P_k be a homogeneous polynomial of degree k. Let us define the Fourier coefficients*

$$a_{j,k} = \frac{1}{\pi} \int_0^{2\pi} P_k(\cos t, \sin t)\, \cos(jt)\, dt \quad \textit{for} \quad j \in \{0, 1 \ldots, k\}$$

and

$$b_{j,k} = \frac{1}{\pi} \int_0^{2\pi} P_k(\cos t, \sin t)\, \sin(jt)\, dt \quad \textit{for} \quad j \in \{1 \ldots, k\}.$$

Then, for every $R > 0$,

$$\mathrm{Cap}_{B(0,R)}(B_\varepsilon, P_k) = \pi\, D(P_k)\, \varepsilon^{2k}(1 + o(1))$$

as $\varepsilon \to 0^+$, where $B_\varepsilon = \overline{B}(0, \varepsilon)$ and $D(P_k)$ is a constant depending only on the coefficients of the polynomial P_k given by

$$D(P_k) = \frac{k\, a_{0,k}^2}{4} + \sum_{j=1}^{k} \frac{(k+j)^2}{2k} (a_{j,k}^2 + b_{j,k}^2). \tag{38}$$

Proof. Let us denote by V the Dirichlet potential of P_k and by W its expression in polar coordinates, that is to say $V(r\cos t, r\sin t) = W(r,t)$ for $(r,t) \in (0,R) \times (0, 2\pi)$. By definition of the Fourier coefficients,

$$P_k(r\cos t, r\sin t) = r^k \frac{a_{0,k}}{2} + \sum_{j=1}^{k} r^k\, (a_{j,k}\cos(jt) + b_{j,k}\sin(jt)) \tag{39}$$

for all $(r,t) \in (0, +\infty) \times (0, 2\pi)$. For all $x \in B_\varepsilon$, $V(x) = P_k(x)$, and therefore, using polar coordinates,

$$\int_{B(0,\varepsilon)} |\nabla V(x)|^2\, dx = \int_0^\varepsilon r^{2k-2} \Bigg[\int_0^{2\pi} k^2 \left(\frac{a_{0,k}}{2} + \sum_{j=1}^{k} a_{j,k}\cos(jt) + b_{j,k}\sin(jt) \right)^2 + \left(\sum_{j=1}^{k} j\, b_{j,k}\cos(jt) - j\, a_{j,k}\sin(jt) \right)^2 \Bigg]\, r\, dr.$$

By Parseval's identity, we obtain

$$\int_{B(0,\varepsilon)} |\nabla V(x)|^2\, dx = \frac{\varepsilon^{2k}}{2k} \left(\frac{k^2\, \pi\, a_{0,k}^2}{2} + \sum_{j=1}^{k} \pi\, (k^2 + j^2)\, (a_{j,k}^2 + b_{j,k}^2) \right). \tag{40}$$

Let us now determine V in the open set $B(0,R) \setminus B_\varepsilon$, that is to say $W(r,t)$ for $r \in (\varepsilon, R)$. The function W satisfy the boundary value problem

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} W\right) + \frac{1}{r^2}\frac{\partial^2}{\partial t^2} W = 0, & \text{in } (\varepsilon, R) \times (0, 2\pi), \\ W(R,t) = 0, & \text{for all } t \in (0, 2\pi), \\ W(\varepsilon, t) = P_k(\varepsilon\cos t, \varepsilon\sin t), & \text{for all } t \in (0, 2\pi), \\ W(r, 0) = W(r, 2\pi), & \text{for all } r \in (\varepsilon, R). \end{cases} \tag{41}$$

To solve problem (41), we expand W in Fourier series with respect to the variable t:

$$W(r,t) = \frac{a_0(r)}{2} + \sum_{j\geq 1} a_j(r)\cos(jt) + b_j(r)\sin(jt),$$

for $(r,t) \in (\varepsilon, R) \times (0, 2\pi)$. Then

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}W\right) + \frac{1}{r^2}\frac{\partial^2}{\partial t^2}W\right)(r,t) = \frac{1}{2r}\left(r\, a_0'(r)\right)'$$
$$+ \sum_{j=1}^{k}\left(\frac{1}{r}\left(r\, a_j'(r)\right)' - \frac{j^2}{r^2}a_j(r)\right)\cos(jt) + \left(\frac{1}{r}\left(r\, b_j'(r)\right)' - \frac{j^2}{r^2}b_j(r)\right)\sin(jt),$$

so that

$$(r\, a_0'(r))' = 0 \quad \text{in } (\varepsilon, R),$$

and, for $j \geq 1$,

$$r\,(r\, a_j'(r))' - j^2 a_j(r) = 0 \quad \text{and} \quad r\,(r\, b_j'(r))' - j^2 b_j(r) = 0.$$

Taking into account the boundary conditions in (41), we find

$$a_0(r) = a_{0,k}\,\varepsilon^k \frac{\log\left(\frac{r}{R}\right)}{\log\left(\frac{\varepsilon}{R}\right)},$$

and, for $j \in \{1, \ldots, k\}$,

$$a_j(r) = a_{j,k}\varepsilon^k \frac{\left(\frac{R}{r}\right)^j - \left(\frac{r}{R}\right)^j}{\left(\frac{R}{\varepsilon}\right)^j - \left(\frac{\varepsilon}{R}\right)^j} \quad \text{and} \quad b_j(r) = b_{j,k}\varepsilon^k \frac{\left(\frac{R}{r}\right)^j - \left(\frac{r}{R}\right)^j}{\left(\frac{R}{\varepsilon}\right)^j - \left(\frac{\varepsilon}{R}\right)^j},$$

while, for $j \geq k+1$, $a_j(r) = 0$ and $b_j(r) = 0$. Using polar coordinates and Parseval's identity as above, we find

$$\int_{B(0,R)\setminus B_\varepsilon} |\nabla V(x)|^2\, dx$$
$$= \int_\varepsilon^R \left(\frac{\pi}{2}|a_0'(r)|^2 + \pi\sum_{j=1}^{k}\left(|a_j'(r)|^2 + \frac{j^2}{r^2}|a_j(r)|^2 + |b_j'(r)|^2 + \frac{j^2}{r^2}|b_j(r)|^2\right)\right) r\, dr.$$

We have

$$\int_\varepsilon^R \frac{\pi}{2}|a_0'(r)|^2\, r\, dr = \frac{\pi\, a_{0,k}^2\,\varepsilon^{2k}}{2\log^2\left(\frac{R}{\varepsilon}\right)}\int_\varepsilon^R \frac{dr}{r} = \frac{\pi\, a_{0,k}^2\,\varepsilon^{2k}}{2\log\left(\frac{R}{\varepsilon}\right)}.$$

For $j \in \{1, \ldots, k\}$, an integration by parts gives us

$$\int_\varepsilon^R \left(r\,|a_j'(r)|^2 + \frac{j^2}{r}|a_j(r)|^2\right) dr = \left[r\, a_j(r)\, a_j'(r)\right]_\varepsilon^R - \int_\varepsilon^R \left(\left(r\, a_j'(r)\right)' - \frac{j^2}{r}a_j(r)\right) a_j(r)\, dr$$
$$= -\varepsilon\, a_j(\varepsilon)\, a_j'(\varepsilon).$$

Using the expression computed for $a_j(r)$, we find

$$\int_\varepsilon^R \left(r\,|a_j'(r)|^2 + \frac{j^2}{r}|a_j(r)|^2\right) dr = j\, a_{j,k}^2\,\varepsilon^{2k}\frac{1 + \left(\frac{\varepsilon}{R}\right)^{2j}}{1 - \left(\frac{\varepsilon}{R}\right)^{2j}}.$$

The exact same computation gives us

$$\int_\varepsilon^R \left(r\,|b_j'(r)|^2 + \frac{j^2}{r}|b_j(r)|^2\right) dr = j\, b_{j,k}^2\,\varepsilon^{2k}\frac{1 + \left(\frac{\varepsilon}{R}\right)^{2j}}{1 - \left(\frac{\varepsilon}{R}\right)^{2j}},$$

so that

$$\int_{B(0,R)\setminus B_\varepsilon} |\nabla V(x)|^2\, dx = \frac{\pi\, a_{0,k}^2\, \varepsilon^{2k}}{2\log\left(\frac{R}{\varepsilon}\right)} + \pi\, \varepsilon^{2k} \sum_{j=1}^{k} j\,(a_{j,k}^2 + b_{j,k}^2)\, \frac{1+\left(\frac{\varepsilon}{R}\right)^{2j}}{1-\left(\frac{\varepsilon}{R}\right)^{2j}}. \tag{42}$$

Combining (40) and (42), we get

$$\mathrm{Cap}_{B(0,R)}(B_\varepsilon, P_k) = \pi\, \varepsilon^{2k} \left(\frac{k\, a_{0,k}^2}{4} + \sum_{j=1}^{k} \left(\frac{k^2+j^2}{2k} + j \right) (a_{j,k}^2 + b_{j,k}^2) + O\left(\frac{1}{|\log(\varepsilon)|} \right) \right),$$

and finally

$$\mathrm{Cap}_{B(0,R)}(B_\varepsilon, P_k) = \pi \left(k\frac{a_{0,k}^2}{4} + \sum_{j=1}^{k} \frac{(k+j)^2}{2k} (a_{j,k}^2 + b_{j,k}^2) \right) \varepsilon^{2k}\,(1+o(1)),$$

as $\varepsilon \to 0^+$. □

Remark 2.9. *Since the polynomial P_k in Lemma 2.8 is of degree $k \geq 1$, it is non zero, and therefore $D(P_k) > 0$.*

We can now find the asymptotics of the P_k-capacity for small balls concentrating at a point in any open set. This is the analogue of Proposition 2.6 for segments.

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with $0 \in \Omega$. For $\varepsilon > 0$ small, let $B_\varepsilon = \overline{B}(0,\varepsilon)$ and P_k be a homogeneous polynomial of degree $k \geq 0$. Then*

$$\mathrm{Cap}_\Omega(B_\varepsilon, P_k) = \begin{cases} \frac{2\pi c_0^2}{|\log\varepsilon|}\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right), & \text{if } k = 0 \text{ and } P_k \equiv c_0, \\ \pi\, D(P_k)\, \varepsilon^{2k}(1+o(1)), & \text{if } k \geq 1, \end{cases} \tag{43}$$

as $\varepsilon \to 0^+$, where $D(P_k)$ is the constant defined in (38).

Proof. If $k = 0$, i.e. if $P_k \equiv c_0$, then $\mathrm{Cap}_\Omega(B_\varepsilon, P_k) = c_0^2 \mathrm{Cap}_\Omega B_\varepsilon$ and the conclusion follows from Proposition 1.6.

For $k \geq 1$, let us fix two radii $0 < R_1 < R_2$ such that $B(0,R_1) \subset \Omega \subset B(0,R_2)$. By monotonicity of the capacity we have

$$\mathrm{Cap}_{B(0,R_2)}(B_\varepsilon, P_k) \leq \mathrm{Cap}_\Omega(B_\varepsilon, P_k) \leq \mathrm{Cap}_{B(0,R_1)}(s_\varepsilon, P_k).$$

We apply Lemma 2.8 to $\mathrm{Cap}_{B(0,R_1)}(B_\varepsilon, P_k)$ and $\mathrm{Cap}_{B(0,R_1)}(B_\varepsilon, P_k)$ and obtain (43). □

The following proposition is the analogue for disks of Proposition 2.7 for segments.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with $0 \in \Omega$ and let $k \in \mathbb{N} \cup \{0\}$. Let us assume that $u \in C^{k+1}_{\mathrm{loc}}(\Omega)\setminus\{0\}$ has vanishing order at 0 equal to k, i.e. the Taylor polynomial of u of order k and center 0 has degree k and is non-zero and k-homogeneous. Then*

$$\mathrm{Cap}_\Omega(B_\varepsilon, u) = \begin{cases} \frac{2\pi}{|\log\varepsilon|}\, u^2(0)\left(1+o(1)\right), & \text{if } k = 0, \\ \pi\, D(P_k)\, \varepsilon^{2k}(1+o(1)), & \text{if } k \geq 1, \end{cases}$$

as $\varepsilon \to 0^+$, $D(P_k)$ being defined in (38).

Proof. The proof follows by repeating the same arguments as in Proposition 2.7 and using Proposition 2.10 instead of Proposition 2.6. □

Proof of Theorems 1.13 and 1.14. Arguing as in the proof of Theorem 1.9, from the fact that $u \in C^\infty(\Omega)$ and (11) we deduce that the Taylor polynomial of the function u with center 0 and order k is harmonic, k-homogeneous, and has degree k; more precisely it has the form

$$P_k(r\cos t, r\sin t) = \beta r^k \sin(\alpha - kt).$$

Then, for $k \geq 1$, the Fourier coefficients $a_{j,k}$ and $b_{j,k}$ appearing in Lemma 2.8 are zero for $j \neq k$ and

$$a_{k,k} = \begin{cases} 2\beta\sin\alpha, & \text{if } k = 0, \\ \beta\sin\alpha, & \text{if } k \geq 1, \end{cases} \quad \text{and} \quad b_{k,k} = -\beta\cos\alpha.$$

From (38) it follows that, for $k \geq 1$, $D(P_k) = 2k\beta^2$. Then the asymptotics stated in (13) follows form Proposition 2.11.

The proof of Theorem 1.14 follows directly from Theorems 1.4 and 1.13. □

3 Asymptotic expansion for coalescing poles of Aharonov-Bohm operators

In this section we study Aharonov-Bohm operators on domains having one axis of symmetry. More specifically, let us define the reflection $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\sigma(x_1, x_2) = (x_1, -x_2),$$

and let us consider Ω, an open, bounded, and connected set in $\mathbb{R}^2$ satisfying $\sigma(\Omega) = \Omega$. Let us consider a Schrödinger operator with a purely magnetic potential of Aharonov-Bohm type, with two poles on the axis of symmetry

$$\mathcal{R} := \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_2 = 0\},$$

each with a half-integer flux.

More precisely, let us fix two points $a^- = (a_1^-, 0)$ and $a^+ = (a_1^+, 0)$ in $\mathcal{R}$, with $a_1^- < a_1^+$. We consider the vector field A_{a^-,a^+} defined on the doubly punctured plane $\ddot{\mathcal{P}}(a^-, a^+) := \mathbb{R}^2 \setminus \{a^-, a^+\}$ by

$$A_{a^-,a^+}(x) := -\frac{1}{2}\frac{1}{(x_1 - a_1^-)^2 + x_2^2}(-x_2, x_1 - a_1^-) + \frac{1}{2}\frac{1}{(x_1 - a_1^+)^2 + x_2^2}(-x_2, x_1 - a_1^+).$$

Let us note that, if we write, for any $x = (x_1, x_2) \in \ddot{\mathcal{P}}(a^-, a^+)$,

$$A_{a^-,a^+}(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2)),$$

we have

$$A_{a^-,a^+}(x_1, -x_2) = (-A_1(x_1, x_2), A_2(x_1, x_2)).$$

Equivalently, we have, for any $x \in \ddot{\mathcal{P}}(a^-, a^+)$,

$$A_{a^-,a^+}(\sigma(x)) = -A_{a^-,a^+}(x)S$$

where S is a 2×2 symmetry matrix:

$$S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We work in the complex Hilbert space $L^2(\Omega)$ of complex valued square integrable functions on Ω, with the scalar product defined by

$$\langle u, v \rangle := \int_\Omega u\bar{v}\,dx$$

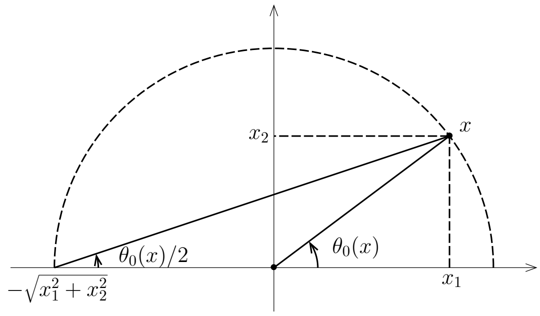


Figure 1: Definition of $\theta_0(x)$.

for u and v in $L^2(\Omega)$. Our operator is the Friedrichs extension of the differential operator

$$(i\nabla + A_{a^-,a^+})^2,$$

acting on $C_c^\infty(\ddot{\Omega}(a^-,a^+))$, the space of smooth functions with compact support in the doubly punctured domain $\ddot{\Omega}(a^-,a^+) := \Omega \setminus \{a^-,a^+\}$. We denote it by $H_{A_{a^-,a^+}}$. It is a positive and self-adjoint operator, with compact resolvent. Its spectrum therefore consists in a sequence of real positive eigenvalues tending to $+\infty$, which we denote by $(\lambda_k(a^-,a^+))_{k\geq 1}$.

3.1 Gauge transformations

We now construct suitable gauge transformations, in order to remove the magnetic potential. We use the notation

$$\mathcal{I}_c(a^-,a^+) := [a_1^-,a_1^+] \times \{0\}$$

to denote the closed segment joining the two poles.

Lemma 3.1. *There exists a unique C^∞-function φ_{a^-,a^+} defined on $\mathbb{R}^2 \setminus \mathcal{I}_c(a^-,a^+)$ such that*

$$\nabla\varphi_{a^-,a^+} = A_{a^-,a^+} \quad \text{on } \mathbb{R}^2 \setminus \mathcal{I}_c(a^-,a^+)$$

and

$$\varphi_{a^-,a^+}(x_1,0) = 0 \text{ for all } x_1 \in (a_1^+,+\infty).$$

Furthermore, φ_{a^-,a^+} satisfies $\varphi_{a^-,a^+} \circ \sigma = -\varphi_{a^-,a^+}$.

Proof. First, we define $\theta_0 : \mathbb{R}^2 \setminus \mathcal{R}_- \to \mathbb{R}$, where $\mathcal{R}_- = (-\infty,0] \times \{0\}$, by

$$\theta_0(x_1,x_2) := 2\arctan\left(\frac{x_2}{x_1+\sqrt{x_1^2+x_2^2}}\right).$$

As seen on Figure 1, $\theta_0(x)$ is the angular coordinate of the point x. The function θ_0 is smooth on $\mathbb{R}^2 \setminus \mathcal{R}_-$, and

$$\nabla\theta_0(x_1,x_2) = \frac{1}{x_1^2+x_2^2}(-x_2,x_1) \qquad (44)$$

for all $x = (x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{R}_-$. Let us note also that $\theta_0 \circ \sigma = -\theta_0$.

Then we define, for any $x = (x_1,x_2) \in \mathbb{R}^2 \setminus ((-\infty,a_1^+] \times \{0\})$,

$$\varphi_{a^-,a^+}(x) := \frac{1}{2}\theta_0(x_1-a_1^+,x_2) - \frac{1}{2}\theta_0(x_1-a_1^-,x_2).$$

See Figure 2 for a geometric interpretation. In particular, if $x_1 > a_1^+$,

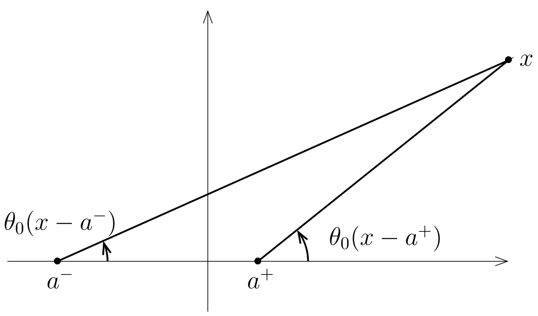


Figure 2: Geometric interpretation of the function φ.

$$\varphi_{a^-,a^+}(x_1,0) = \frac{1}{2}\theta_0(x_1 - a_1^+, 0) - \frac{1}{2}\theta_0(x_1 - a_1^-, 0) = 0 - 0 = 0.$$

The function φ_{a^-,a^+} is smooth on $\mathbb{R}^2 \setminus ((-\infty, a_1^+] \times \{0\})$. Furthermore (44), together with the chain rule, gives

$$\begin{aligned}\nabla\varphi_{a^-,a^+}(x) &= \frac{1}{2}\frac{1}{(x_1-a_1^+)^2+x_2^2}(-x_2, x_1 - a_1^+) - \frac{1}{2}\frac{1}{(x_1-a_1^-)^2+x_2^2}(-x_2, x_1 - a_1^-)\\ &= A_{a^-,a^+}(x)\end{aligned}$$

for all $x \in \mathbb{R}^2 \setminus (-\infty, a_1^+] \times \{0\}$. Furthermore we have that, for any $x_1 < 0$,

$$\lim_{\eta\to 0, \eta>0} \theta_0(x_1, \eta) = \pi \quad \text{and} \quad \lim_{\eta\to 0, \eta>0} \theta_0(x_1, -\eta) = -\pi.$$

This implies that

$$\lim_{\eta\to 0, \eta>0} \varphi_{a^-,a^+}(x_1, \eta) = \lim_{\eta\to 0, \eta>0} \varphi_{a^-,a^+}(x_1, -\eta) = 0$$

for all $x_1 < a_1^-$ and that

$$\lim_{\eta\to 0, \eta>0} \varphi_{a^-,a^+}(x_1, \eta) = \frac{\pi}{2} \quad \text{and} \quad \lim_{\eta\to 0, \eta>0} \varphi_{a^-,a^+}(x_1, -\eta) = -\frac{\pi}{2}$$

for all $x_1 \in (a_1^-, a_1^+)$. In particular, φ_{a^-,a^+} has a continuous extension to $\mathbb{R}^2 \setminus \mathcal{I}_c(a^-, a^+)$, which we also denote by φ_{a^-,a^+}. Since A_{a^-,a^+} is smooth on $\mathbb{R}^2 \setminus \mathcal{I}_c(a^-, a^+)$, this extension is also smooth and satisfies $\nabla\varphi_{a^-,a^+} = A_{a^-,a^+}$. Since $\mathbb{R}^2 \setminus \mathcal{I}_c(a_1^-, a_1^+)$ is connected, the uniqueness is obvious. Furthermore, since $\theta_0 \circ \sigma = -\theta_0$, we have that $\varphi_{a^-,a^+} \circ \sigma = -\varphi_{a^-,a^+}$. □

Lemma 3.2. *There exists a unique smooth function* $\psi_{a^-,a^+} : \ddot{\mathcal{P}}(a^-, a^+) \to \mathbb{C}$ *satisfying*

(i) $|\psi_{a^-,a^+}| \equiv 1$ *on* $\ddot{\mathcal{P}}(a^-, a^+)$*;*

(ii) $\dfrac{\nabla\psi_{a^-,a^+}}{i\psi_{a^-,a^+}} = 2A_{a^-,a^+}$ *on* $\ddot{\mathcal{P}}(a^-, a^+)$*;*

(iii) $\psi_{a^-,a^+}(x_1, 0) = -1$ *for all* $x_1 \in (a_1^-, a_1^+)$*.*

Furthermore, ψ_{a^-,a^+} *satisfies* $\psi_{a^-,a^+} \circ \sigma = \overline{\psi}_{a^-,a^+}$*.*

Proof. For all $x \in \mathbb{R}^2 \setminus \mathcal{I}_c(a^-, a^+)$, we set $\psi_{a^-,a^+}(x) = e^{2i\varphi_{a^-,a^+}(x)}$, where φ_{a^-,a^+} is the function defined in Lemma 3.1. This function is smooth on $\mathbb{R}^2 \setminus \mathcal{I}_c(a^-, a^+)$ and, for all $x \in \mathbb{R}^2 \setminus \mathcal{I}_c(a^-, a^+)$,

$$\nabla\psi_{a^-,a^+}(x) = 2ie^{2i\varphi_{a^-,a^+}(x)}\nabla\varphi_{a^-,a^+}(x) = 2i\psi_{a^-,a^+}(x)A_{a^-,a^+}(x),$$

and thus

$$\frac{\nabla\psi_{a^-,a^+}(x)}{i\psi_{a^-,a^+}(x)} = 2\nabla\varphi_{a^-,a^+}(x) = 2A_{a^-,a^+}(x).$$

On the other hand, for all $x_1 \in (a_1^-, a_1^+)$,

$$\lim_{\eta\to 0,\eta>0}\psi_{a^-,a^+}(x_1,\eta) = e^{i\pi} = -1 \text{ and } \lim_{\eta\to 0,\eta>0}\psi_{a^-,a^+}(x_1,-\eta) = e^{-i\pi} = -1.$$

This implies that ψ_{a^-,a^+} admits a continuous extension to $\ddot{\mathcal{P}}(a^-,a^+)$, which we also denote by ψ_{a^-,a^+}. Since $\nabla\psi_{a^-,a^+} = 2i\psi_{a^-,a^+}A_{a^-,a^+}$ on $\mathbb{R}^2\setminus\mathcal{I}_c(a^-,a^+)$, with A_{a^-,a^+} smooth on $\ddot{\mathcal{P}}(a^-,a^+)$, we obtain that ψ_{a^-,a^+} is of class C^1 on $\ddot{\mathcal{P}}(a^-,a^+)$, and then that ψ_{a^-,a^+} is smooth by a bootstrap argument.

Let us now prove uniqueness. Let us assume that $\widetilde{\psi}$ is a function satisfying conditions (i-iii). Then, we deduce from (ii) that

$$\nabla\left(\frac{\widetilde{\psi}}{\psi_{a^-,a^+}}\right) = \frac{\widetilde{\psi}}{\psi_{a^-,a^+}}\left(\frac{\nabla\widetilde{\psi}}{\widetilde{\psi}} - \frac{\nabla\psi_{a^-,a^+}}{\psi_{a^-,a^+}}\right) = \frac{\widetilde{\psi}}{\psi_{a^-,a^+}}\left(2iA_{a^-,a^+} - 2iA_{a^-,a^+}\right) = 0$$

on $\ddot{\mathcal{P}}(a^-,a^+)$. There exists therefore $c\in\mathbb{C}$ such that $\widetilde{\psi} = c\psi_{a^-,a^+}$ on $\ddot{\mathcal{P}}(a^-,a^+)$, and condition (iii) tells us that $c=1$, thus proving uniqueness.

Finally, since $\varphi_{a^-,a^+}\circ\sigma = -\varphi_{a^-,a^+}$, we conclude that $\psi_{a^-,a^+}\circ\sigma = \overline{\psi_{a^-,a^+}}$. □

To simplify notation, in the following sections, we do not write explicitly the dependence on a^- and a^+, except for the eigenvalues, but the objects considered depend on the position of the two poles (so we will write H_A for $H_{A_{a^-,a^+}}$, $\ddot{\Omega}$ for $\ddot{\Omega}(a^-,a^+)$, etc.).

3.2 Conjugation and symmetry

Definition 3.3. *Let us define the antilinear, antiunitary operator K on $L^2(\Omega)$ by $Ku := \psi\overline{u}$, where ψ is the gauge function defined in Lemma 3.2. We say that a function $u\in L^2(\Omega)$ is* K-real *if $Ku = u$. We denote by $L^2_K(\Omega)$ the set of K-real functions.*

Lemma 3.4. *If u and v are in $L^2_K(\Omega)$, $\langle u,v\rangle\in\mathbb{R}$.*

Proof. We have

$$\langle u,v\rangle = \int_\Omega u\overline{v}\,dx = \int_\Omega \overline{\psi}u\psi\overline{v}\,dx = \int_\Omega \overline{\psi\overline{u}}\psi\overline{v}\,dx = \int_\Omega \overline{u}v\,dx = \overline{\langle u,v\rangle}.$$ □

Remark 3.5. *The set $L^2_K(\Omega)$ is not a subspace of the complex vector space $L^2(\Omega)$, because multiplication by a complex number does not preserve K-real functions. However, multiplication by a real number does preserve these functions, and therefore $L^2_K(\Omega)$ is a real vector space. Moreover, Lemma 3.4 shows that the restriction to $L^2_K(\Omega)$ of the complex scalar product on $L^2(\Omega)$ is a real scalar product. Therefore, $L^2_K(\Omega)$ is a real Hilbert space.*

Lemma 3.6. *The antilinear operator K preserves the domain of H_A, and $H_A\circ K = K\circ H_A$.*

Proof. Let us begin by considering $u\in C_c^\infty(\ddot{\Omega})$. We have

$$(i\nabla + A)(Ku) = (i\nabla + A)(\psi\overline{u}) = i\psi\nabla\overline{u} + \psi\overline{u}A + i\overline{u}\nabla\psi.$$

Since $\nabla\psi = 2i\psi A$, we obtain

$$(i\nabla + A)(Ku) = i\psi\nabla\overline{u} - \psi\overline{u}A = -\psi\overline{(i\nabla + A)u}.$$

As a consequence, for any $v \in C_c^\infty(\ddot{\Omega})$,

$$\begin{aligned}\int_\Omega H_A K u\, \overline{v}\, dx &= \int_\Omega (i\nabla + A)(Ku)\cdot \overline{(i\nabla + A)v}\, dx = -\int_\Omega \psi \overline{(i\nabla + A)u}\cdot \overline{(i\nabla + A)v}\, dx\\ &= \int_\Omega (i\nabla + A)(Kv)\cdot \overline{(i\nabla + A)u}\, dx = \int_\Omega Kv\, \overline{(i\nabla + A)^2 u}\, dx\\ &= \int_\Omega \psi \overline{v}\, \overline{(i\nabla + A)^2 u}\, dx = \int_\Omega K H_A u\, \overline{v}\, dx.\end{aligned}$$

We therefore have $H_A K u = K H_A u$ for all $u \in C_c^\infty(\ddot{\Omega})$. The conclusion follows by density. □

We deduce from Lemma 3.6 that the eigenspaces of H_A are stable under the action of K. This implies that we can find a basis of $L^2(\Omega)$ formed by K-real eigenfunctions of H_A. We also interpret this in another way: $L^2_K(\Omega)$ is stable under the action of H_A and the restriction of H_A to $L^2_K(\Omega)$ has the same spectrum as H_A.

We now want to study the consequence of the fact that Ω is symmetric with respect to $\mathcal{R}$ on the operator H_A. We therefore define the antiunitary antilinear operator Σ^c, acting on $L^2(\Omega)$, by $\Sigma^c u := \overline{u}\circ\sigma$.

Lemma 3.7. *The antilinear operator Σ^c preserves the domain of H_A, and $H_A \circ \Sigma^c = \Sigma^c \circ H_A$. Furthermore, $\Sigma^c \circ K = K \circ \Sigma^c$.*

Proof. The second point is clear: if $u \in L^2(\Omega)$,

$$(\Sigma^c K)\, u(x) = \overline{(Ku)\,(\sigma(x))} = \overline{\psi(\sigma(x))\overline{u(\sigma(x))}} = \psi(x)\overline{\overline{u(\sigma(x))}} = (K\Sigma^c)u(x).$$

To prove the first point, we begin by considering $u \in C_c^\infty(\ddot{\Omega})$. We have

$$(i\nabla + A)(\Sigma^c u) = i\nabla(\overline{u}\circ\sigma) + (\overline{u})\circ\sigma A = (i(\nabla\overline{u})\circ\sigma)S - ((\overline{u}A)\circ\sigma)S = -\overline{((i\nabla + A)u\circ\sigma)}S.$$

For any $v \in C_c^\infty(\ddot{\Omega})$, we have

$$\int_\Omega (H_A\Sigma^c u)\, \overline{v}\, dx = \int_\Omega (i\nabla + A)(\Sigma^c u)\cdot \overline{(i\nabla + A)v}\, dx = \int_\Omega -\overline{((i\nabla + A)u\circ\sigma)}S\cdot \overline{(i\nabla + A)v}\, dx.$$

After the change of variable $x = \sigma(y)$, and using the fact that S is symmetric, we find

$$\begin{aligned}\int_\Omega (H_A\Sigma^c u)\, \overline{v}\, dx &= \int_\Omega \overline{(i\nabla + A)u}\cdot -\overline{((i\nabla + A)v\circ\sigma)}S\, dy\\ &= \int_\Omega \overline{(i\nabla + A)u}\cdot (i\nabla + A)(\Sigma^c v)\, dy = \int_\Omega \overline{H_A u}\, \Sigma^c v\, dy = \int_\Omega \overline{H_A u}\, (\overline{v}\circ\sigma)\, dy.\end{aligned}$$

We now do the reverse change of variable $y = \sigma(x)$, thus obtaining

$$\int_\Omega (H_A\Sigma^c u)\, \overline{v}\, dx = \int_\Omega \left(\overline{H_A u}\circ\sigma\right)\overline{v}\, dx = \int_\Omega (\Sigma^c H_A u)\, \overline{v}\, dx.$$

We therefore have $H_A\Sigma^c u = \Sigma^c H_A u$ for all $u \in C_c^\infty(\ddot{\Omega})$. The conclusion follows by density. □

The second point of Lemma 3.7 implies that $L^2_K(\Omega)$ is stable under the action of Σ^c. If we write

$$L^2_{K,\Sigma}(\Omega) := L^2_K(\Omega)\cap \ker(\Sigma^c - Id)$$

and

$$L^2_{K,a\Sigma}(\Omega) := L^2_K(\Omega)\cap \ker(\Sigma^c + Id),$$

we observe that every function $u \in L^2_K(\Omega)$ can be decomposed as

$$u = \frac{1}{2}(u + \bar{u}\circ\sigma) + \frac{1}{2}(u - \bar{u}\circ\sigma), \tag{45}$$

so that we have the orthogonal decomposition

$$L^2_K(\Omega) = L^2_{K,\Sigma}(\Omega) \oplus L^2_{K,a\Sigma}(\Omega). \tag{46}$$

The first point of Lemma 3.7 implies that H_A leaves the spaces $L^2_{K,\Sigma}$ and $L^2_{K,a\Sigma}$ invariant. We can therefore define the operators $H_{A,\Sigma}$ and $H_{A,a\Sigma}$, restrictions of H_A to $L^2_{K,\Sigma}(\Omega)$ and $L^2_{K,a\Sigma}(\Omega)$ respectively. The spectrum of H_A is the reunion (counted with multiplicities) of the spectra of $H_{A,\Sigma}$ and $H_{A,a\Sigma}$.

3.3 Spectral equivalence to the Laplacian with mixed boundary conditions

Let us consider the *real* Hilbert space $L^2_{\mathbb{R},\sigma}(\Omega)$ consisting of the real valued L^2-functions u on Ω such that $u \circ \sigma = u$.

Let us consider the operator H_{NDN} on $L^2_{\mathbb{R},\sigma}(\Omega)$ defined as the Friedrichs extension of the differential operator $-\Delta$ acting on the domain $\{u \in C^\infty_c(\Omega \setminus \mathcal{I}_c, \mathbb{R}) : u \circ \sigma = u\}$, the space of real valued smooth functions with compact support in $\Omega \setminus \mathcal{I}_c$ symmetric with respect to the axis $x_2 = 0$. The domain of H_{NDN} is then given by $\{u \in H^1_0(\Omega \setminus \mathcal{I}_c) : u \circ \sigma = u \text{ and } \Delta\big|_{\Omega\setminus\mathcal{I}_c} u \in L^2_{\mathbb{R},\sigma}(\Omega)\}$, being $\Delta\big|_{\Omega\setminus\mathcal{I}_c}$ the distributional Laplacian in $\Omega \setminus \mathcal{I}_c$. H_{NDN} is a symmetric, positive, and self-adjoint operator on $L^2_{\mathbb{R},\sigma}(\Omega)$. We denote by $(\lambda^{NDN}_k(a^+, a^-))_{k\geq 1}$ its eigenvalues.

In a similar way, we consider the operator H_{DND} on $L^2_{\mathbb{R},\sigma}(\Omega)$ defined as the Friedrichs extension of $-\Delta$ acting on $\{u \in C^\infty_c(\Omega \setminus (\mathcal{R} \setminus \mathcal{I}_c), \mathbb{R}) : u \circ \sigma = u\}$. The domain of H_{DND} is then given by $\{u \in H^1_0(\Omega \setminus (\mathcal{R} \setminus \mathcal{I}_c)) : u \circ \sigma = u \text{ and } \Delta\big|_{\Omega\setminus(\mathcal{R}\setminus\mathcal{I}_c)} u \in L^2_{\mathbb{R},\sigma}(\Omega)\}$, being $\Delta\big|_{\Omega\setminus(\mathcal{R}\setminus\mathcal{I}_c)}$ the distributional Laplacian in $\Omega \setminus (\mathcal{R} \setminus \mathcal{I}_c)$. H_{DND} is a symmetric, positive, and self-adjoint operator on $L^2_{\mathbb{R},\sigma}(\Omega)$. We denote by $(\lambda^{DND}_k(a^+, a^-))_{k\geq 1}$ its eigenvalues.

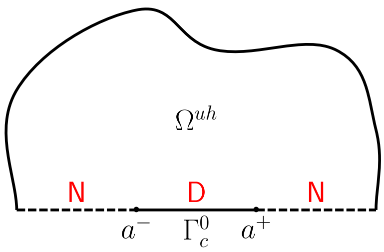
Remark 3.8. *Let us consider the upper-half domain associated with* Ω

$$\Omega^{uh} := \Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}.$$

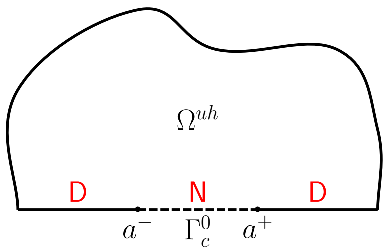
We have that $\partial\Omega^{uh} := \Gamma^{uh} \cup \Gamma_0$, *with* $\Gamma^{uh} := \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 > 0\}$ *and* $\Gamma^0 := \overline{\Omega} \cap \mathcal{R}$. *We additionally define* $\Gamma^0_c = \Gamma^0 \cap \mathcal{I}_c$.

We notice that, if Ω *has smooth boundary, then the operator* H_{NDN} *can be identified with the Neumann-Dirichlet-Neumann Laplacian on* Ω^{uh} *denoted by* $-\Delta^{NDN}$ *and defined as the Laplacian on* Ω^{uh} *with Dirichlet boundary condition on* $\Gamma^{uh} \cup \Gamma^0_c$ *and Neumann boundary condition on* $\Gamma^0 \setminus \Gamma^0_c$, *see Figure 3(a).*

In a similar way, if Ω *has smooth boundary, then the operator* H_{DND} *can be identified with the Dirichlet-Neumann-Dirichlet Laplacian* $-\Delta^{DND}$ *defined as the Laplacian on* Ω^{uh} *with Dirichlet boundary condition on* $\Gamma^{uh} \cup (\Gamma^0 \setminus \Gamma^0_c)$ *and Neumann boundary condition on* Γ^0_c, *see Figure 3(b).*



(a) Neumann-Dirichlet-Neumann boundary conditions



(b) Dirichlet-Neumann-Dirichlet boundary conditions

Figure 3: Eigenvalue problems with mixed boundary conditions in Ω^{uh}.

The main result of this section if the following equivalence of H_{NDN} with $H_{A,\Sigma}$ and of H_{DND} with $H_{A,a\Sigma}$.

Proposition 3.9. *The operator H_{NDN} is unitarily equivalent to $H_{A,\Sigma}$ and the operator H_{DND} is unitarily equivalent to $H_{A,a\Sigma}$.*

Before proving Proposition 3.9, we observe that a direct consequence of Proposition 3.9 combined with the discussion in §3.2 is the following isospectrality result.

Corollary 3.10. *The sequence $(\lambda_k(a^+,a^-))$ is the reunion, counted with multiplicities, of the sequences $(\lambda_k^{NDN}(a^+,a^-))_{k\geq 1}$ and $(\lambda_k^{DND}(a^+,a^-))_{k\geq 1}$.*

We divide the proof of Proposition 3.9 into two lemmas. The first gives information on the nodal set of functions in $L^2_{K,\Sigma}(\Omega)$ or $L^2_{K,a\Sigma}(\Omega)$.

Lemma 3.11. *If $u\in L^2_{K,\Sigma}(\Omega)\cap C(\ddot{\Omega})$, then $u\equiv 0$ on $\Omega\cap\mathcal{I}_c$. If $u\in L^2_{K,a\Sigma}(\Omega)\cap C(\ddot{\Omega})$, then $u\equiv 0$ on $(\Omega\cap\mathcal{R})\setminus\mathcal{I}_c$.*

Proof. Since $Ku=u$, we have $\overline{u}(x_1,0)=u(x_1,0)$ if $x_1<a_1^-$ of $x_1>a_1^+$, and $\overline{u}(x_1,0)=-u(x_1,0)$ and if $x_1\in(a_1^-,a_1^+)$.

If $\Sigma^c u=u$, $\overline{u}(x_1,0)=u(x_1,0)$ for all x_1, and therefore $u(x_1,0)=0$ if $x_1\in(a_1^-,a_1^+)$. In the case where $\Sigma^c u=-u$, $\overline{u}(x_1,0)=-u(x_1,0)$ for all x_1, and therefore $u(x_1,0)=0$ if $x_1<a_1^-$ or $x_1>a_1^+$. ☐

The second Lemma gives a unitary operator of similarity, and proves the two isospectrality results of Proposition 3.9.

Lemma 3.12. *If $u\in L^2(\Omega)$, we define*

$$U_\sigma u := e^{-i\varphi}u, \quad U_{a\sigma}u := \begin{cases} e^{-i\varphi}u, & \text{in } \Omega^{uh},\\ -e^{-i\varphi}u, & \text{in } \Omega\setminus\Omega^{uh}\end{cases}$$

where φ is the function defined in Lemma 3.1. We have the following properties:

(i) *U_σ defines a one-to-one and unitary mapping from $L^2_{K,\Sigma}(\Omega)$ to $L^2_{\mathbb{R},\sigma}(\Omega)$ and $U_{a\sigma}$ defines a one-to-one and unitary mapping from $L^2_{K,a\Sigma}(\Omega)$ to $L^2_{\mathbb{R},\sigma}(\Omega)$;*

(ii) *U_σ maps the domain of $H_{A,\Sigma}$ to the domain of H_{NDN} and $U_\sigma\circ H_{A,\Sigma}=H_{NDN}\circ U_\sigma$;*

(iii) *$U_{a\sigma}$ maps the domain of $H_{A,a\Sigma}$ to the domain of H_{DND} and $U_{a\sigma}\circ H_{A,a\Sigma}=H_{DND}\circ U_{a\sigma}$.*

Proof. Let us first check that for all $u\in L^2_K(\Omega)$, $\overline{U_\sigma u}=U_\sigma u$ and $\overline{U_{a\sigma}u}=U_{a,\sigma}u$. Indeed, for $x\in\Omega$,

$$\overline{e^{-i\varphi(x)}u(x)}=e^{-i\varphi(x)}e^{2i\varphi(x)}\overline{u}(x)=e^{-i\varphi(x)}(Ku)(x)=e^{-i\varphi(x)}u(x).$$

If $u\in L^2_{K,\Sigma}(\Omega)$, then $U_\sigma u(\sigma(x))=e^{-i\varphi(\sigma(x))}u(\sigma(x))=e^{i\varphi(x)}\overline{u(x)}=\overline{U_\sigma u(x)}=U_\sigma u(x)$ so that $U_\sigma u\in L^2_{\mathbb{R},\sigma}(\Omega)$. If $u\in L^2_{K,a\Sigma}(\Omega)$, then $U_{a\sigma}u(\sigma(x))=U_{a\sigma}u(x)$ so that $U_{a\sigma}u\in L^2_{\mathbb{R},\sigma}(\Omega)$.

Furthermore, if $u\in L^2_K(\Omega)$ then $|U_\sigma u|=|u|$ and $|U_{a\sigma}u|=|u|$, therefore

$$\int_\Omega |u|^2\,dx=\int_\Omega |U_\sigma u|^2\,dx=\int_\Omega |U_{a\sigma}u|^2\,dx.$$

Finally, if $v\in L^2_{\mathbb{R},\sigma}(\Omega)$, a direct computation shows that the function u_Σ defined on Ω by

$$u_\Sigma(x):=e^{i\varphi(x)}v(x)$$

is in $L^2_{K,\Sigma}(\Omega)$ and that $U_\sigma u_\Sigma=v$. This shows that U_σ defines a one-to-one map from $L^2_{K,\Sigma}(\Omega)$ to $L^2_{\mathbb{R},\sigma}(\Omega)$. In the same way, the function $u_{a\Sigma}$ defined on Ω by

$$u_{a\Sigma}(x):=e^{i\varphi(x)}v(x) \text{ for } x\in\Omega^{uh}$$

and

$$u_{a\Sigma}(x) := -e^{i\varphi(x)}v(x) \text{ for } x \in \Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 < 0\}$$

is in $L^2_{K,a\Sigma}(\Omega)$ and $U_{a\sigma}u_{a\Sigma} = v$. This shows that $U_{a\sigma}$ defines a one-to-one map from $L^2_{K,a\Sigma}(\Omega)$ to $L^2_{\mathbb{R},\sigma}(\Omega)$. We have proved point (i).

To prove point (ii), let us begin by considering $u \in C_c^\infty(\ddot{\Omega}) \cap L^2_{K,\Sigma}(\Omega)$. According to Lemma 3.11, $u \equiv 0$ on $\Omega \cap \mathcal{I}_c$. Then $U_\sigma u \in H^1_0(\Omega \setminus \mathcal{I}_c)$ (for this it is crucial that u vanishes on $\mathcal{I}_c$ since $e^{-i\varphi}$ jumps across $\mathcal{I}_c$) and

$$(i\nabla)(U_\sigma u) = i\nabla(e^{-i\varphi}u) = e^{-i\varphi}(i\nabla + \nabla\varphi)u = e^{-i\varphi}(i\nabla + A)u \quad \text{in } \Omega \setminus \mathcal{I}_c.$$

We observe that any function u in the domain of the operator $H_{A,\Sigma}$ can be approximated in the form domain norm by functions in $C_c^\infty(\ddot{\Omega}) \cap L^2_{K,\Sigma}(\Omega)$. To this aim, we can first take a sequence of functions $u_n \in C_c^\infty(\ddot{\Omega})$ converging to u in the form domain norm; then we take the sequence $v_n = \frac{1}{4}(u_n + \overline{u_n \circ \sigma} + K(u_n + \overline{u_n \circ \sigma})$ which stays in $C_c^\infty(\ddot{\Omega}) \cap L^2_{K,\Sigma}(\Omega)$ and converges to u in the form domain norm thanks to the validity of the Hardy type inequality

$$\|Aw\|_{L^2(\Omega)} \leq C(a, \Omega)\|(i\nabla + A)w\|_{L^2(\Omega)}$$

which holds for every $w \in C_c^\infty(\ddot{\Omega})$ and for some $C(a, \Omega) > 0$ depending on a and Ω.

Then we conclude that, for every u in the domain of the operator $H_{A,\Sigma}$, $U_\sigma u \in H^1_0(\Omega \setminus \mathcal{I}_c)$ and

$$(i\nabla)(U_\sigma u) = e^{-i\varphi}(i\nabla + A)u \quad \text{in } \Omega \setminus \mathcal{I}_c.$$

Furthermore, for every $w \in C_c^\infty(\Omega \setminus \mathcal{I}_c)$

$$\int_{\Omega\setminus\mathcal{I}_c} \nabla(U_\sigma u) \cdot \nabla w \, dx = \int_{\ddot{\Omega}} (i\nabla + A)u \cdot \overline{(i\nabla + A)(e^{i\varphi}w)} \, dx. \tag{47}$$

Since u is in the domain of the Friedrichs extension of the differential operator $(i\nabla + A)^2$, we conclude that

$$\left| \int_{\Omega\setminus\mathcal{I}_c} \nabla(U_\sigma u) \cdot \nabla w \, dx \right| \leq \text{const}\|w\|_{L^2(\Omega)}$$

for every $w \in C_c^\infty(\Omega \setminus \mathcal{I}_c)$, thus implying that $U_\sigma u$ stays in the domain of H_{NDN}. Moreover, by density and (47) we conclude that

$$U_\sigma(H_{NDN}u) = e^{-i\varphi}(i\nabla + A)^2 u$$

completing the proof of (ii).

The proof of (iii) can be obtained in a similar way, observing that any $u \in C_c^\infty(\ddot{\Omega}) \cap L^2_{A,a\Sigma}(\Omega)$ vanishes on $\mathcal{R} \setminus \mathcal{I}_c$; hence $U_{a\sigma}u \in H^1_0(\Omega \setminus (\mathcal{R} \setminus \mathcal{I}_c))$ (for this it is crucial that u vanishes on $\mathcal{R} \setminus \mathcal{I}_c$ since $\text{sign}(x_2)e^{-i\varphi}$ jumps across $\mathcal{R} \setminus \mathcal{I}_c$). □

3.4 Proof of Theorem 1.16

Combining the isospectrality result of Corollary 3.10 with Theorem 1.10 we can now prove Theorem 1.16.

Proof of Theorem 1.16. It is known from [16] that $\lambda_N^a \to \lambda_N(\Omega)$; in particular the continuity result of [16] implies that, since $\lambda_N(\Omega)$ is simple, then also λ_N^a is simple. It is not restrictive to assume that u_N is real valued. It is easy to prove that, for all $a > 0$ small, there exists u_N^a eigenfunction of $(i\nabla + A_{a^-,a^+})^2$ associate to λ_N^a such that

$$u_N^a \to u_N \quad \text{in } C^2_{\text{loc}}(\Omega \setminus \{0\}, \mathbb{C}) \tag{48}$$

as $a \to 0^+$; moreover it is possible to choose $u_N^a \in L^2_{K_{a^-,a^+}}(\Omega)$ (otherwise take $\frac{1}{2}(u_N^a + K_{a^-,a^+}(u_N^a))$ which is a K_{a^-,a^+}-real eigenfunction for λ_N^a still converging to u_N).

The orthogonal decomposition (45) and the simplicity of λ_N^a imply that either $u_N^a \in L^2_{K,\Sigma}(\Omega)$ (and then, by Lemma 3.11, $u_N^a \equiv 0$ on $[-a,a] \times \{0\}$) or $u_N^a \in L^2_{K,a\Sigma}(\Omega)$ (and then $u_N^a \equiv 0$ on $(\mathbb{R} \setminus (-a,a)) \times \{0\}$). If $u_N^a \equiv 0$ on $(\mathbb{R} \setminus (-a,a)) \times \{0\}$, then (48) would imply that $u_N \equiv 0$ on $\mathbb{R} \times \{0\}$ thus contradicting the assumption $\alpha \neq 0$. Hence we have that necessarily $u_N^a \in L^2_{K,\Sigma}(\Omega)$. Then λ_N^a is an eigenvalue of $H_{A_{a^-,a^+},\Sigma}$ and, by Proposition 3.9, of H_{NDN}. Therefore

$$\lambda_N^a = \lambda_N(\Omega \setminus ([-a,a] \times \{0\}))$$

and the conclusion follows applying Theorem 1.10. □

A Proof of Theorem 1.4

Since our setting is a little different from [12] (which considers manifolds without boundary) and Theorem 1.4 is quite hidden in the arguments of [12], we think it is worthwhile giving in this appendix a proof of Theorem 1.4. Our approach is different from the one used in [12]. It relies on the spectral theorem to estimate how closely approximate eigenvalues and eigenfunctions approach the true one.

Let us begin with the following crucial estimate, which is the analogue of [12, Lemma 3.2].

Lemma A.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, and open set. If $(K_\varepsilon)_{\varepsilon>0}$ is a family of compact sets contained in Ω concentrating to a compact set K with $\mathrm{Cap}_\Omega K = 0$, then for every $f \in H_0^1(\Omega)$*

$$\int_\Omega |V_{K_\varepsilon,f}|^2\,dx = o\left(\mathrm{Cap}_\Omega(K_\varepsilon, f)\right) \qquad \text{as } \varepsilon \to 0.$$

Proof. Let us assume by contradiction that there exists a sequence $\varepsilon_n \to 0$ and a constant $C > 0$ such that

$$\int_\Omega |V_{K_{\varepsilon_n},f}|^2\,dx \geq \frac{1}{C}\mathrm{Cap}_\Omega(K_{\varepsilon_n}, f).$$

We set

$$W_n := \frac{1}{\left\|V_{K_{\varepsilon_n},f}\right\|_{L^2(\Omega)}} V_{K_{\varepsilon_n},f}.$$

We have

$$\|W_n\|_{L^2(\Omega)} = 1$$

and

$$\|\nabla W_n\|^2_{L^2(\Omega)} = \frac{1}{\left\|V_{K_{\varepsilon_n},f}\right\|^2_{L^2(\Omega)}}\mathrm{Cap}_\Omega(K_{\varepsilon_n}, f) \leq C.$$

By weak compactness of the unit ball of $H_0^1(\Omega)$ and compactness of the inclusion $H_0^1(\Omega) \subset L^2(\Omega)$, there exists an increasing sequence of integers $(n_k)_{k\geq1}$ and a function $W \in H_0^1(\Omega)$ such that $(W_{n_k})_{k\geq1}$ converges to W when k goes to $+\infty$, weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. We have that $\|W\|_{L^2(\Omega)} = 1$ and $\Delta W = 0$ in $\Omega \setminus K$ in a weak sense. This last equation implies that W is harmonic in Ω (since $\mathrm{Cap}_\Omega K = 0$), and therefore that W is identically 0. We have reached a contradiction and proved the lemma. □

Corollary A.2. *If $(K_\varepsilon)_{\varepsilon>0}$ is a family of compact sets contained in Ω concentrating to a compact set $K \subset \Omega$ with $\mathrm{Cap}_\Omega K = 0$, then, for any $f \in H_0^1(\Omega) \cap L^\infty(\Omega)$,*

$$\int_\Omega |V_{K_\varepsilon,f}|^2\,dx = o\left(\mathrm{Cap}_\Omega(K_\varepsilon)\right) \qquad \text{as } \varepsilon \to 0.$$

Proof. By the maximum principle for harmonic functions in $\Omega \setminus K_\varepsilon$ we have that

$$|V_{K_\varepsilon,f}| \le \Big(\max_{\Omega} |f|\Big)\, V_{K_\varepsilon}.$$

Hence $\int_\Omega |V_{K_\varepsilon,f}|^2\,dx \le (\max_\Omega |f|)^2 \int_\Omega |V_{K_\varepsilon}|^2 dx$ and the conclusion follows from Lemma A.1 (with $f = \eta_K$). ☐

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. For $\varepsilon > 0$, we denote by $-\Delta_\varepsilon$ the Dirichlet Laplacian on $\Omega \setminus K_\varepsilon$. More precisely, $-\Delta_\varepsilon$ is the self-adjoint operator obtained from the restriction of the quadratic form

$$q(u) = \int_\Omega |\nabla u|^2\, dx$$

to $H^1_0(\Omega \setminus K_\varepsilon)$ through the Friedrichs' extension procedure (see for instance [20, Theorem X.23]).

To simplify notation, we write $\lambda_\varepsilon = \lambda_N(\Omega \setminus K_\varepsilon)$, $c_\varepsilon = \mathrm{Cap}_\Omega(K_\varepsilon, u_N)$, $V_\varepsilon = V_{K_\varepsilon,u_N}$, and we denote by q both the quadratic form defined above and the associated bilinear form. We write $\psi_\varepsilon = u_N - V_\varepsilon$. Let us note that by definition of the potential V_ε, ψ_ε is the orthogonal projection of u_N on $H^1_0(\Omega \setminus K_\varepsilon)$, in the space $H^1_0(\Omega)$ endowed with the scalar product q. For any $\varphi \in H^1_0(\Omega \setminus K_\varepsilon)$,

$$\begin{aligned} q(\psi_\varepsilon,\varphi) - \lambda_N(\Omega)\langle \psi_\varepsilon,\varphi\rangle_{L^2(\Omega)} &= q(u_N,\varphi) - \lambda_N(\Omega)\langle \psi_\varepsilon,\varphi\rangle_{L^2(\Omega)} \\ &= \lambda_N(\Omega)\langle u_N,\varphi\rangle_{L^2(\Omega)} - \lambda_N(\Omega)\langle \psi_\varepsilon,\varphi\rangle_{L^2(\Omega)} = \lambda_N(\Omega)\langle V_\varepsilon,\varphi\rangle_{L^2(\Omega)}. \end{aligned}$$

This means that ψ_ε is in the domain of the operator $-\Delta_\varepsilon$ and that

$$(-\Delta_\varepsilon - \lambda_N(\Omega))\psi_\varepsilon = \lambda_N(\Omega) V_\varepsilon. \tag{49}$$

According to Lemma A.1, $\|V_\varepsilon\|_{L^2(\Omega)} = o\big(c_\varepsilon^{1/2}\big)$ as $\varepsilon \to 0^+$, so that

$$\|(-\Delta_\varepsilon - \lambda_N(\Omega))\psi_\varepsilon\|_{L^2(\Omega)} = o\big(c_\varepsilon^{1/2}\big)$$

as $\varepsilon \to 0^+$. From the spectral theorem (see for instance [15, Proposition 8.20]), we get

$$\mathrm{dist}\,(\lambda_N(\Omega), \sigma(-\Delta_\varepsilon)) \le \frac{\|(-\Delta_\varepsilon - \lambda_N(\Omega))\psi_\varepsilon\|_{L^2(\Omega)}}{\|\psi_\varepsilon\|_{L^2(\Omega)}} = o\big(c_\varepsilon^{1/2}\big), \quad \text{as } \varepsilon \to 0^+,$$

where $\sigma(-\Delta_\varepsilon)$ is the spectrum of the self-adjoint operator $-\Delta_\varepsilon$. We recall that $\lambda_\varepsilon \to \lambda_N(\Omega)$ as $\varepsilon \to 0^+$: this an immediate corollary of [19, Theorem 2.3]. Since $\lambda_N(\Omega)$ is assumed to be simple, λ_ε is simple for $\varepsilon > 0$ small enough, and

$$|\lambda_\varepsilon - \lambda_N(\Omega)| = o\big(c_\varepsilon^{1/2}\big) \quad \text{as } \varepsilon \to 0^+.$$

Let us now denote by Π_ε the orthogonal projection from $L^2(\Omega)$ onto the one-dimensional eigenspace associated with λ_ε, and let us write $\widetilde{u}_\varepsilon := \psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon$. We have

$$(-\Delta_\varepsilon - \lambda_\varepsilon)\,\Pi_\varepsilon\psi_\varepsilon = 0,$$

and therefore

$$(-\Delta_\varepsilon - \lambda_\varepsilon)\,\widetilde{u}_\varepsilon = (-\Delta_\varepsilon - \lambda_\varepsilon)\,\psi_\varepsilon.$$

Since

$$\|(-\Delta_\varepsilon - \lambda_\varepsilon)\,\psi_\varepsilon\|_{L^2(\Omega)} \le |\lambda_N(\Omega) - \lambda_\varepsilon|\,\|\psi_\varepsilon\|_{L^2(\Omega)} + \|(-\Delta_\varepsilon - \lambda_N(\Omega))\psi_\varepsilon\|_{L^2(\Omega)}\,,$$

we obtain

$$\|(-\Delta_\varepsilon - \lambda_\varepsilon)\,\widetilde{u}_\varepsilon\|_{L^2(\Omega)} = o\big(c_\varepsilon^{1/2}\big) \quad \text{as } \varepsilon \to 0^+.$$

Let us denote by $\mathcal{K}_\varepsilon$ the closed subspace $\mathrm{Im}(I - \Pi_\varepsilon) = \ker(\Pi_\varepsilon)$, and by T_ε the restriction of the operator $-\Delta_\varepsilon$ to $\mathcal{K}_\varepsilon$. The operator T_ε is self-adjoint, with spectrum $\sigma(T_\varepsilon) = \sigma(-\Delta_\varepsilon) \setminus \{\lambda_\varepsilon\}$. Furthermore, since $\lambda_j(\Omega \setminus K_\varepsilon) \to \lambda_j(\Omega)$ for all $j \in \mathbb{N}_1$ as $\varepsilon \to 0^+$, and since $\lambda_N(\Omega)$ is simple, there exists some $\delta > 0$ such that $\mathrm{dist}(\lambda_\varepsilon, \sigma(T_\varepsilon)) \geq \delta$ for $\varepsilon > 0$ small enough. Using the spectral theorem for the operator T_ε, we get

$$\mathrm{dist}\,(\lambda_\varepsilon, \sigma(T_\varepsilon))\, \|\widetilde{u}_\varepsilon\|_{L^2(\Omega)} \leq \|(T_\varepsilon - \lambda_\varepsilon)\, \widetilde{u}_\varepsilon\|_{L^2(\Omega)}\,,$$

and therefore

$$\|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)} \leq \frac{\|(T_\varepsilon - \lambda_\varepsilon)\, \widetilde{u}_\varepsilon\|_{L^2(\Omega)}}{\delta} = o\big(c_\varepsilon^{1/2}\big) \quad \text{as } \varepsilon \to 0^+.$$

Consequently, we have

$$\|u_N - \Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)} \leq \|V_\varepsilon\|_{L^2(\Omega)} + \|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\| = o\big(c_\varepsilon^{1/2}\big),$$

and therefore

$$\|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)} = 1 + o\big(c_\varepsilon^{1/2}\big) \quad \text{as } \varepsilon \to 0^+.$$

This implies in particular that $\Pi_\varepsilon \psi_\varepsilon$ is non-zero for $\varepsilon > 0$ small enough, so that we can define

$$u_\varepsilon = \frac{\Pi_\varepsilon \psi_\varepsilon}{\|\Pi_\varepsilon \psi_\varepsilon\|_{L^2(\Omega)}},$$

an $L^2(\Omega)$-normalized eigenfunction of $-\Delta_\varepsilon$ associated with λ_ε. A simple computation shows that

$$\|u_\varepsilon - \psi_\varepsilon\|_{L^2(\Omega)} = o\big(c_\varepsilon^{1/2}\big)$$

and

$$\|u_\varepsilon - u_N\|_{L^2(\Omega)} = o\big(c_\varepsilon^{1/2}\big)$$

as $\varepsilon \to 0^+$. Taking the scalar product of equation (49) with u_ε, we obtain

$$\begin{aligned}(\lambda_\varepsilon - \lambda_N(\Omega))\langle u_\varepsilon, \psi_\varepsilon\rangle_{L^2(\Omega)} &= \lambda_N(\Omega)\langle u_N, V_\varepsilon\rangle_{L^2(\Omega)} + \lambda_N(\Omega)\langle u_\varepsilon - u_N, V_\varepsilon\rangle_{L^2(\Omega)} &(50)\\ &= \lambda_N(\Omega)\langle u_N, V_\varepsilon\rangle_{L^2(\Omega)} + o\,(c_\varepsilon) \quad \text{as } \varepsilon \to 0^+. &(51)\end{aligned}$$

On the other hand, since ψ_ε and V_ε are q-orthogonal, we have

$$c(\varepsilon) = q(V_\varepsilon) = q(u_N - \psi_\varepsilon, V_\varepsilon) = q(u_N, V_\varepsilon) = \lambda_N(\Omega)\langle u_N, V_\varepsilon\rangle_{L^2(\Omega)}, \tag{52}$$

using for the last equality the fact that u_N is an eigenfunction of the quadratic form q, associated with the eigenvalue $\lambda_N(\Omega)$. Reinjecting (52) into (50), we finally obtain

$$\lambda_\varepsilon - \lambda_N(\Omega) = \frac{c_\varepsilon + o(c_\varepsilon)}{\langle u_\varepsilon, \psi_\varepsilon\rangle} = c_\varepsilon(1 + o(1)),$$

as $\varepsilon \to 0^+$. □

B Continuity of the u-capacity

In this appendix, we establish a continuity result for the u-capacity with respect to concentration at zero capacity sets.

Proposition B.1. *If $\{K_\varepsilon\}_{\varepsilon>0}$ is a family of compact sets contained in $\Omega \subset \mathbb{R}^n$ concentrating to a compact set $K \subset \Omega$ with $\mathrm{Cap}_\Omega(K) = 0$, then, for every $u \in H^1_0(\Omega)$, we have that $V_{K_\varepsilon,u} \to V_{K,u} = 0$ strongly in $H^1_0(\Omega)$ and $\lim_{\varepsilon\to 0^+} \mathrm{Cap}_\Omega(K_\varepsilon, u) = \mathrm{Cap}_\Omega(K, u) = 0$.*

Proof. Testing equation (9) for $V_{K_\varepsilon,u}$ with $\varphi = V_{K_\varepsilon,u} - u$ we obtain

$$\begin{aligned} 0 &= \int_{\Omega\setminus K_\varepsilon} \nabla V_{K_\varepsilon,u}\cdot\nabla(V_{K_\varepsilon,u}-u)\,dx \\ &= \int_\Omega \nabla V_{K_\varepsilon,u}\cdot\nabla(V_{K_\varepsilon,u}-u)\,dx = \int_\Omega |\nabla V_{K_\varepsilon,u}|^2\,dx - \int_\Omega \nabla V_{K_\varepsilon,u}\cdot\nabla u\,dx. \end{aligned} \tag{53}$$

Since $V_{K_\varepsilon,u}$ attains the minimum defining $\mathrm{Cap}_\Omega(K_\varepsilon,u)$, we have that $\int_\Omega |\nabla V_{K_\varepsilon,u}|^2\,dx \le \int_\Omega |\nabla u|^2\,dx$, so that $\{V_{K_\varepsilon,u}\}_{\varepsilon>0}$ is bounded in $H^1_0(\Omega)$. Hence, along a sequence $\varepsilon_k\to 0^+$, $V_{K_{\varepsilon_k},u}\rightharpoonup V$ weakly in $H^1_0(\Omega)$ for some $V\in H^1_0(\Omega)$. Since $\mathrm{Cap}_\Omega(K)=0$, we have that $H^1_0(\Omega)=H^1_0(\Omega\setminus K)$ (see [12, Proposition 2.1]), hence $u-V\in H^1_0(\Omega\setminus K)$. Moreover, for every $\varphi\in C^\infty_c(\Omega\setminus K)$, we have that $\varphi\in C^\infty_c(\Omega\setminus K_\varepsilon)$ for ε sufficiently small, hence, passing to the limit in (9) for $V_{K_{\varepsilon_k},u}$ as $k\to+\infty$, we obtain that

$$\int_\Omega \nabla V\cdot\nabla\varphi\,dx = 0.$$

Hence $\int_\Omega \nabla V\cdot\nabla\varphi\,dx = 0$ for every $\varphi\in H^1_0(\Omega\setminus K)=H^1_0(\Omega)$. It follows that $V=V_{K,u}=0$. Moreover, passing to the limit in (53), we obtain that

$$\lim_{k\to+\infty}\mathrm{Cap}_\Omega(K_{\varepsilon_k},u) = \lim_{k\to+\infty}\int_\Omega |\nabla V_{K_{\varepsilon_k},u}|^2\,dx = \lim_{k\to+\infty}\int_\Omega \nabla V_{K_{\varepsilon_k},u}\cdot\nabla u\,dx = \int_\Omega \nabla V\cdot\nabla u\,dx = 0$$

We conclude that $\mathrm{Cap}_\Omega(K_{\varepsilon_k},u)\to 0$ and $V_{K_{\varepsilon_k},u}\to 0$ strongly in $H^1_0(\Omega)$ as $k\to+\infty$. Since such limits do not depend on the subsequence, we reach the conclusion. □

Acknowledgements The authors are partially supported by the project ERC Advanced Grant 2013 n. 339958: "Complex Patterns for Strongly Interacting Dynamical Systems – COMPAT". V. Felli is partially supported by PRIN-2012-grant "Variational and perturbative aspects of nonlinear differential problems".

References

[1] L. Abatangelo and V. Felli. Sharp asymptotic estimates for eigenvalues of Aharonov-Bohm operators with varying poles. *Calc. Var. Partial Differential Equations*, 54(4):3857–3903, 2015.

[2] L. Abatangelo and V. Felli. On the leading term of the eigenvalue variation for Aharonov-Bohm operators with a moving pole. *SIAM J. Math. Anal.*, 48(4):2843–2868, 2016.

[3] L. Abatangelo, V. Felli, and C. Léna. On Aharonov-Bohm operators with two colliding poles. *Preprint 2016.*

[4] L. Abatangelo, V. Felli, B. Noris, and M. Nys. Sharp boundary behavior of eigenvalues for Aharonov-Bohm operators with varying poles. `arXiv:1605.09569`.

[5] J. Bertrand and B. Colbois. Capacité et ineégaliteé de FaberKrahn dans $\mathbb{R}^n$. *Journal of Functional Analysis*, 232(1):1–28, 2006.

[6] G. Besson. Comportement asymptotique des valeurs propres du Laplacien dans un domaine avec un trou. *Bull. Soc. Math. France*, 113(2):211–230, 1985.

[7] V. Bonnaillie-Noël, B. Helffer, and T. Hoffmann-Ostenhof. Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions. *J. Phys. A*, 42, 185203, 2009.

[8] V. Bonnaillie-Noël, B. Noris, M. Nys, and S. Terracini. On the eigenvalues of Aharonov–Bohm operators with varying poles. *Analysis & PDE*, 7(6):1365–1395, 2014.

[9] G. M. Capriani. The Steiner rearrangement in any codimension. *Calc. Var. Partial Differential Equations*, 49(1):517–548, 2014.

[10] I. Chavel and E. Feldman. Spectra of manifolds less a small domain. *Duke Math. J.*, 56(2):399–414, 1988.

[11] Y. Colin de Verdière. Sur la multiplicité de la première valeur propre non nulle du Laplacien. *Comment. Math. Helv.*, 61(2):254–270, 1986.

[12] Gilles Courtois. Spectrum of manifolds with holes. *J. Funct. Anal.*, 134(1):194–221, 1995.

[13] V. Felli, A. Ferrero, and S. Terracini. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *Journal of the European Mathematical Society*, 13(1):119–174, 2011.

[14] M. Flucher. Approximation of Dirichlet eigenvalues on domains with small holes. *J. Math. Anal. Appl.*, 193(1):169–199, 1995.

[15] Bernard Helffer. *Spectral theory and its applications*, volume 139 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[16] C. Léna, Eigenvalues variations for Aharonov-Bohm operators. *Journal of Mathematical Physics* 56, 011502, 2015.

[17] Noris, B., M. Nys, and S. Terracini. On the Aharonov–Bohm operators with varying poles: The boundary behavior of eigenvalues. *Communications in Mathematical Physics*, 339(3):1101–1146, 2015.

[18] S. Ozawa. Singular variation of domains and eigenvalues of the Laplacian. *Duke Math. J.*, 48(4):767–778, 1981.

[19] J. Rauch and M. Taylor. Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.*, 18, 27–59, 1975.

[20] M. Reed and B. Simon. *Methods of Modern Mathematical physics. II. Fourier Analysis, Self-Adjointness.* Academic Press, New York, 1975.

[21] D.D. Sokolov (originator). Elliptic coordinates. Encyclopedia of Mathematics, http://www.encyclopediaofmath.org/index.php/Elliptic_coordinates, 2011. Accessed: 2015-07-27.