Generalizations of Pleijel's nodal domain theorem

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Introduction

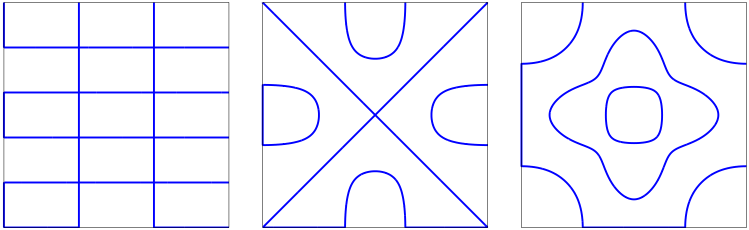
We study eigenvalue problems in a domain Ω, that is to say an open, bounded and connected subset of $\mathbb{R}^n$, or more generally of an n-dimensional Riemannian manifold. The eigenvalue equation is

$$-\Delta u = \lambda u \text{ in } \Omega$$

with a boundary condition on $\partial\Omega$: $u = 0$ (Dirichlet), $\frac{\partial u}{\partial n} = 0$ (Neumann) or $\frac{\partial u}{\partial n} + hu = 0$ (Robin).
For an eigenfunction u, we define the nodal set as

$$\mathcal{N}(u) := \overline{u^{-1}(\{0\})}$$

and the nodal domains as the connected components of $\Omega \setminus \mathcal{N}(u)$.



Some classical results

Theorem (J. C. F. Sturm [1836])

For a regular Sturm-Liouville problem on a segment (a, b), with eigenvalues $(\lambda_k)_{k \geq 1}$, an eigenfunction associated with λ_k has $k - 1$ zeros in (a, b).

Remark

The theorem of Sturm and Liouville states that a k-th eigenfunction divides the segment into k nodal domains.

Theorem (R. Courant [1923])

Let Ω be an open, bounded and connected set in $\mathbb{R}^n$, with a sufficiently regular boundary. Let $(\lambda_k)_{k \geq 1}$ be the eigenvalues of the Laplacian, with Dirichlet, Neumann or Robin boundary condition. Then, an eigenfunction associated with λ_k has at most k nodal domains.

Pleijel's result

Let Ω be an open, bounded and connected set in $\mathbb{R}^2$, and let $(\lambda_k)_{k\geq 1}$ denote the eigenvalue of the Laplacian with a Dirichlet boundary condition.

Theorem (Å. Pleijel [1956])

There is only a finite number of indices k for which λ_k has an eigenfunction with k nodal domains.

Let ν_k denote the maximal number of nodal domains for an eigenfunction associated with λ_k.

Proposition (Å. Pleijel [1956])

We have the asymptotic upper bound

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^2}{\lambda_1(\mathbb{D})\,|\mathbb{D}|^2} = \frac{4}{j^2} < 1.$$

The n-dimensional case

The result can be generalized to $\mathbb{R}^n$. We keep the same notation.

Theorem

If $\Omega \subset \mathbb{R}^n$,

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^{\frac{n}{2}} \, |\mathbb{B}^n|^2} < 1.$$

We write

$$\gamma(n) := \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^{\frac{n}{2}} \omega_n^2}.$$

We have the explicit expression

$$\gamma(n) = \frac{2^{n-2} n^2 \Gamma\left(\frac{n}{2}\right)^2}{j_{\frac{n}{2}-1,1}^n},$$

where $j_{\frac{n}{2}-1,1}$ is the smallest positive zero of the Bessel function of the first kind $J_{\frac{n}{2}-1}$.

Theorem (B. Helffer, M. Persson Sundqvist [2016])

The sequence $n \mapsto \gamma(n)$ is decreasing and goes to 0 exponentially fast $(\gamma(n+1)/\gamma(n) \to 2/e)$.

Outline of the proof

Let u be an eigenfunction associated with λ_k with ν_k nodal domains. Let us denote by $D_1, \ldots, D_{\nu_k}$ the nodal domains of u.
We apply the Faber-Krahn inequality to a domain D_i:

$$\lambda_k^{\frac{n}{2}} \, |D_i| = \lambda_1(D_i)^{\frac{n}{2}} \, |D_i| \geq \lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n.$$

Summing over $i \in \{1, \ldots, \nu_k\}$, we obtain

$$\lambda_k^{\frac{n}{2}} |\Omega| \geq \nu_k \, \lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n$$

and therefore

$$\nu_k \leq \frac{\lambda_k^{\frac{n}{2}} |\Omega|}{\lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n}.$$

According to Weyl's law

$$\lambda_k^{\frac{n}{2}} |\Omega| \sim \frac{(2\pi)^n k}{\omega_n}.$$

We obtain

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)^{\frac{n}{2}} \omega_n^2}.$$

Some extensions

The asymptotic upper bound

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n)$$

holds for the eigenfunctions of the following operators.

- The Laplace-Beltrami operator in Ω, with a Dirichlet boundary condition on $\partial\Omega$, where Ω is an open and connected set compactly included in a 2-dimensional Riemannian manifold M ($n = 2$), with M homeomorphic to a disk (J. Petree [1957]).
- The Laplace-Beltrami operator in M, a compact n-dimensional Riemannian manifold with or without boundary, with a Dirichlet boundary condition on ∂M if it is not empty (P. Bérard and D. Meyer [1982]).
- The Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$, for several choices of V, including the (possibly anisotropic) harmonic potential and the Coulomb potential (P. Charron [2015] and P. Charron, B. Helffer and T. Hoffmann-Ostenhof [2016]).

Neumann boundary condition: an example

Given $\Omega \subset \mathbb{R}^2$ an open, bounded and connected set with a sufficiently regular boundary, we consider the Laplacian in Ω with Neumann boundary condition, and denote by $(\mu_k(\Omega))_{k\geq 1}$ its eigenvalues.

Proposition (Å. Pleijel [1956])

If $\Omega = Q := (0,\pi)^2$,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.$$

Proof.

Given an eigenfunction u, divide its nodal domains into

- the interior domains $D_1^0, \dots, D_{\nu_k^0}^0$ not touching ∂Q;
- the boundary domains $D_1^1, \dots, D_{\nu_k^1}^1$ adjacent to ∂Q.

Using the fact that the eigenfunctions, restricted to one of the four sides of Q, are trigonometric polynomials of degree at most $\sqrt{\mu_k(Q)}$, we get $\nu_k^1 \leq C\sqrt{\mu_k(Q)}$ for some constant C, so that $\lim_{k\to+\infty} \frac{\nu_k^1}{k} = 0$.
On the other hand, we can bound the number of interior domains as in the Dirichlet case. □

Neumann boundary condition: generalization

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a piecewise analytic boundary. We again consider the Laplacian in Ω with Neumann boundary condition, whose eigenvalues we denote by $(\mu_k)_{k\geq 1}$.

Theorem (J.A. Toth, S. Zelditch [2009])

For $k \geq 1$, we denote by r_k the greatest possible number of zeros of u on $\partial\Omega$, where u is an eigenfunction associated with μ_k. There exists a constant C_Ω such that

$$r_k \leq C_\Omega \sqrt{\mu_k}.$$

Theorem (I. Polterovich [2009])

Under the above hypotheses for Ω, for the Neumann-Laplacian eigenfunctions,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.$$

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Statement of the result

Let Ω be an open, bounded and connected open set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{1,1}$.
For $h \in \mathrm{Lip}\left(\overline{\Omega}\right)$ such that $h \geq 0$ on $\partial\Omega$, we consider the eigenvalue problem with Robin boundary condition

$$\begin{cases} -\Delta\, u & = & \mu\, u & \text{in } \Omega; \\ \frac{\partial u}{\partial n} + h\, u & = & 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by $(\mu_k)_{k\geq 1}$ the associated sequence of eigenvalues, arranged in non-decreasing order and counted with multiplicities.
Using standard regularity result for elliptic boundary value problems, we can show that any eigenfunction of the above problem is of class $C^1\left(\overline{\Omega}\right)$.
As before, we denote by ν_k the maximal number of nodal domain for an eigenfunction associated with μ_k.

Theorem

We have the asymptotic upper bound

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma(n).$$

Preliminaries: inequalities in a nodal domain

Let (μ, u) is an eigenpair and D a nodal domain of u. We have

$$\int_D |\nabla u|^2 \leq \mu \int_D u^2.$$

Indeed,

$$\int_D |\nabla u|^2 = \int_D (-\Delta u)\, u + \int_{\partial D} \frac{\partial u}{\partial n} u = \mu \int_D u^2 - \int_{\partial D} h\, u^2 =$$
$$\mu \int_D u^2 - \int_{\partial D \cap \Omega} h\, u^2 - \int_{\partial D \cap \partial\Omega} h\, u^2 \leq \mu \int_D u^2.$$

Furthermore, the Faber-Krahn inequality gives a lower bound for the Rayleigh quotients. More precisely, for each $v \in H_0^1(D)$, we define

$$R(v, D) = \frac{\int_D |\nabla v|^2}{\int_D v^2},$$

and we have

$$\lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n \leq R(v, D)^{\frac{n}{2}} |D|.$$

The function v does not need to be a groundstate of $-\Delta$ in D.

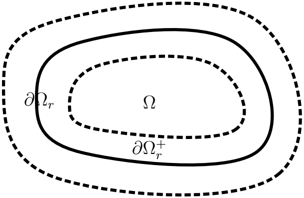
Given a nodal domain D, the main idea is to distinguish between two cases:

- most of the mass of the eigenfunction is inside Ω;
- there is a non-negligible amount of mass near $\partial\Omega$.

This is a natural distinction when we try to apply the previous form of the Faber-Krahn inequality.

Given $r > 0$, we consider

$$\begin{aligned} \partial\Omega_r &:= \{x \in \mathbb{R}^n\,;\, \mathrm{dist}(x, \partial\Omega) < r\}; \\ \partial\Omega_r^+ &:= \partial\Omega_r \cap \Omega. \end{aligned}$$



For $\delta > 0$ small enough, we construct non-negative smooth functions φ_0 and φ_1 such that

- $\varphi_0^2 + \varphi_1^2 = 1$ in Ω,
- $\mathrm{supp}(\varphi_0) \subset \Omega \setminus \overline{\partial\Omega_{a\delta}^+}$ and $\mathrm{supp}(\varphi_1) \subset \partial\Omega_{A\delta}^+$,
- $\|\nabla\varphi_i\|_{L^\infty} \leq C\delta^{-1}$ for $i \in \{0, 1\}$,

with $0 < a < A$ and C independent of δ.

Let us consider an eigenpair (μ, u). We define $u_0 := \varphi_0 u$ and $u_1 := \varphi_1 u$.
By construction of (φ_0, φ_1), we have, for each nodal domain D of u,

$$\int_D u^2 = \int_D u_0^2 + \int_D u_1^2.$$

We fix $\varepsilon \in (0,1)$. With respect to this choice, we say that D is

- a bulk domain if $\int_D u_0^2 \geq (1-\varepsilon) \int_D u^2$;
- a boundary domain if $\int_D u_1^2 > \varepsilon \int_D u^2$.

We write

- $\nu^0(u,\varepsilon)$ for the number of bulk domains;
- $\nu^1(u,\varepsilon)$ for the number of boundary domains.

Ultimately, we want to show that $\nu^1(u,\varepsilon) \ll \nu^0(u,\varepsilon)$ for μ large. We therefore take δ depending on μ, namely

$$\delta := \mu^{-\theta}$$

with $\theta > 0$ to be determined.
Following the steps of Pleijel's proof, we obtain, for the number of bulk domains,

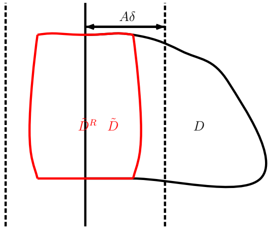
$$\nu^0(u,\varepsilon) \leq \frac{|\Omega|}{\lambda_1(\mathbb{B}^n)^{\frac{n}{2}}\,\omega_n} \left(\frac{1+\varepsilon}{1-\varepsilon}\mu + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon} C^2 \mu^{2\theta} \right)^{\frac{n}{2}}.$$

Given a boundary domain D, we define

$$\widetilde{D} := D \cap \{u_1 \neq 0\} \subset \partial\Omega^+_{A\delta},$$

u_1^R and $\widetilde{D}^R$ the reflection of u_1 and $\widetilde{D}$ through $\partial\Omega$. We have in particular

$$\widetilde{D}^R \subset \partial\Omega_{A\delta}.$$



We have

$$R\left(u_1^R, \widetilde{D}^R\right) = R\left(u_1, \widetilde{D}\right) = \frac{\int_D |\nabla u_1|^2}{\int_D u_1^2} \leq \frac{2}{\varepsilon}\left(\mu + C^2\mu^{2\theta}\right),$$

and, Faber-Krahn inequality applied to $\widetilde{D}^R$ gives us

$$\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n \leq \left|R\left(u_1^R, \widetilde{D}^R\right)\right|^{\frac{n}{2}}\left|\widetilde{D}^R\right| = 2\left|R\left(u_1, \widetilde{D}\right)\right|^{\frac{n}{2}}\left|\widetilde{D}\right|$$

Summing over all boundary domains, we get

$$\nu^1(u,\varepsilon) \leq C' \frac{|\partial\Omega_{A\delta}|}{\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n}\left(\mu + C^2\mu^{2\theta}\right)^{\frac{n}{2}},$$

and therefore

$$\nu^1(u,\varepsilon) \leq C''\mu^{-\theta}\left(\mu + C^2\mu^{2\theta}\right)^{\frac{n}{2}}.$$

Let $(u_k)_{k\geq 1}$ be a sequence of eigenfunction associated with (μ_k) and having each the maximal number of nodal domain, ν_k.
Let us recall that we have

$$\nu^0(u_k,\varepsilon) \leq \frac{|\Omega|}{\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n}\left(\frac{1+\varepsilon}{1-\varepsilon}\mu_k + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon}C^2\mu_k^{2\theta}\right)^{\frac{n}{2}}.$$

and

$$\nu^1(u_k,\varepsilon) \leq C''\mu^{-\theta}\left(\mu_k + C^2\mu_k^{2\theta}\right)^{\frac{n}{2}}.$$

We choose $\theta \in (0, \frac{1}{2})$, for instance $\theta = \frac{1}{4}$. Using Weyl's law, we have

$$\mu_k \leq \lambda_k \sim \frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

Therefore $\lim_{k\to+\infty}\frac{\nu^1(u_k,\varepsilon)}{k} = 0$ and $\limsup_{k\to+\infty}\frac{\nu^0(u_k,\varepsilon)}{k} \leq \frac{(2\,\pi)^n}{\lambda\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n^2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n}{2}}.$

Since $\nu_k = \nu^0(u_k,\varepsilon) + \nu^1(u_k,\varepsilon)$, we get

$$\limsup_{k\to+\infty}\frac{\nu_k}{k} \leq \frac{(2\,\pi)^n}{\lambda\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n^2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n}{2}},$$

and the conclusion when $\varepsilon \to 0$.

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Recent improvements: sharper upper bound

We now go back to eigenvalues of the Laplacian with a Dirichlet boundary condition.

$\Omega \subset \mathbb{R}^2$	$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma(2) - 3.10^{-9}$	(J. Bourgain [2013])
$\Omega \subset \mathbb{R}^2$	$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma(2) - \varepsilon(2)$	(S. Steinerberger [2013])
$\Omega \subset M^n$	$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma(n) - \varepsilon(n)$	(H. Donnelly [2014])

Conjecture (I. Polterovich [2009])

For any domain $\Omega \subset \mathbb{R}^2$,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{2}{\pi}.$$

The analysis of rectangles show that the conjecture upper bound is optimal.

Recent improvements: geometric control

General form of the results
(P. Bérard and B. Helffer [2016], M. van den Berg and K. Gittins [2016])

Given a "regular enough" domain Ω, there are no more than $N(\Omega)$ eigenvalues satisfying equality in Courant Theorem, where $N(\Omega)$ depends on known geometric quantities associated with Ω.

The proofs rely on explicit geometric estimates for the remainder in Weyl's law.

Theorem (M. van den Berg and K. Gittins [2016])

Let Ω be a convex domain in $\mathbb{R}^n$, then there is an (explicit) constant $C(n)$ such that

$$N(\Omega) \leq C(n) \frac{\mathcal{H}^{n-1}(\partial\Omega)^n}{|\Omega|^{n-1}}$$

Can we prove similar results for Neumann/Robin eigenfunctions?