Generalizations of Pleijel's nodal domain theorem

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Introduction

We study eigenvalue problems in a domain Ω, that is to say an open, bounded and connected subset of $\mathbb{R}^n$, or more generally of an n-dimensional Riemannian manifold. The eigenvalue equation is

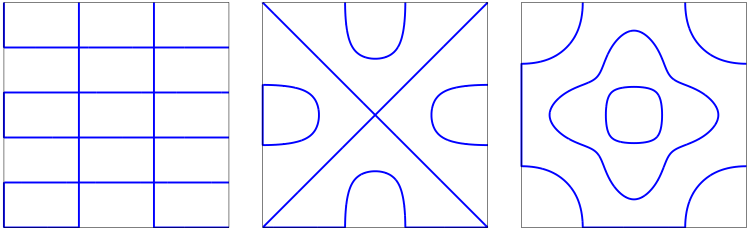
$$-\Delta u = \lambda u \text{ in } \Omega$$

with a boundary condition on $\partial\Omega$: $u = 0$ (Dirichlet), $\frac{\partial u}{\partial n} = 0$ (Neumann) or $\frac{\partial u}{\partial n} + hu = 0$ (Robin).

For an eigenfunction u, we define the nodal set as

$$\mathcal{N}(u) := \overline{u^{-1}(\{0\})}$$

and the nodal domains as the connected components of $\Omega \setminus \mathcal{N}(u)$.



Some classical results

Theorem (J. Liouville, J. C. F. Sturm)

For a regular Sturm-Liouville problem on a segment (a, b), with eigenvalues $(\lambda_k)_{k\geq 1}$, an eigenfunction associated with λ_k has $k-1$ zeros in (a, b).

Remark

The theorem of Sturm and Liouville states that a k-th eigenfunction divides the segment into k nodal domains.

Theorem (R. Courant, 1923)

Let Ω be an open, bounded and connected set in $\mathbb{R}^n$, with a sufficiently regular boundary. Let $(\lambda_k)_{k\geq 1}$ be the eigenvalues of the Laplacian, with Dirichlet, Neumann or Robin boundary condition. Then, an eigenfunction associated with λ_k as at most k nodal domains.

Pleijel's result

Let Ω be an open, bounded and connected set in $\mathbb{R}^2$, and let $(\lambda_k)_{k\geq 1}$ denote the eigenvalue of the Laplacian with a Dirichlet boundary condition.

Theorem (Å. Pleijel, 1956)

There is only a finite number of indices k for which λ_k as an eigenfunction with k nodal domains.

Let ν_k denote the maximal number of nodal domains for an eigenfunction associated with λ_k.

Proposition (Å. Pleijel, 1956)

We have the asymptotic upper bound

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^2}{\lambda_1(\mathbb{D})\,|\mathbb{D}|^2} = \frac{4}{j^2} < 1.$$

The n-dimensional case

The result can be generalized to $\mathbb{R}^n$. We keep the same notation.

Theorem

If $\Omega \subset \mathbb{R}^n$,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)\,|\mathbb{B}^n|^2} < 1.$$

We write

$$\gamma(n) := \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)\omega_n^2}.$$

We have the explicit expression

$$\gamma(n) = \frac{2^{n-2} n^2 \Gamma\left(\frac{n}{2}\right)^2}{j_{\frac{n}{2}-1,1}^2},$$

where $j_{\frac{n}{2}-1,1}$ is the smallest positive zero of the Bessel function of the first kind $J_{\frac{n}{2}-1}$.

Theorem (B. Helffer, M. Persson Sundqvist, 2016)

The sequence $n \mapsto \gamma(n)$ is decreasing and goes to 0 exponentially fast $(\gamma(n+1)/\gamma(n) \to 2/e)$.

Outline of the proof

Let u be an eigenfunction associated with λ_k with ν_k nodal domains. Let us denote by $D_1, \ldots, D_{\nu_k}$ the nodal domains of u.
We apply the Faber-Krahn inequality to a domain D_i:

$$\lambda_k^{\frac{n}{2}} \, |D_i| = \lambda_1(D_i)^{\frac{n}{2}} \, |D_i| \geq \lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n.$$

Summing over $i \in \{1, \ldots, \nu_k\}$, we obtain

$$\lambda_k^{\frac{n}{2}} |\Omega| \geq \nu_k \, \lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n$$

and therefore

$$\nu_k \leq \frac{\lambda_k^{\frac{n}{2}} |\Omega|}{\lambda_1 \left(\mathbb{B}^n\right)^{\frac{n}{2}} \omega_n}.$$

According to Weyl's law

$$\lambda_k^{\frac{n}{2}} |\Omega| \sim \frac{(2\pi)^n k}{\omega_n}.$$

We obtain

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{(2\pi)^n}{\lambda_1(\mathbb{B}^n)\omega_n^2}.$$

Some extensions

The asymptotic upper bound

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \gamma(n)$$

holds for the eigenfunctions of the following operators.

- The Laplace-Beltrami operator in Ω, with a Dirichlet boundary condition on $\partial\Omega$, where Ω is an open and connected set compactly included in a 2-dimensional Riemannian manifold M ($n = 2$), with M homeomorphic to a disk (J. Petree [1957]).
- The Laplace-Beltrami operator in M, a compact n-dimensional Riemannian manifold with or without boundary, with a Dirichlet boundary condition on ∂M if it is not empty (P. Bérard and D. Meyer [1982]).
- The Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$, for several choices of V, including the (possibly anisotropic) harmonic potential and the Coulomb potential (P. Charron [2015] and P. Charron, B. Helffer and T. Hoffmann-Ostenhof [2016]).

Neumann boundary condition: an example

Given $\Omega \subset \mathbb{R}^2$ an open, bounded and connected set with a sufficiently regular boundary, we consider the Laplacian in Ω with Neumann boundary condition, and denote by $(\mu_k(\Omega))_{k\geq 1}$ its eigenvalues.

Proposition (Å. Pleijel [1956])

If $\Omega = Q := (0,\pi)^2$,

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.$$

Proof.

Given an eigenfunction u, divide its nodal domains into

- the interior domains $D_1^0, \dots, D_{\nu_k^0}^0$ not touching ∂Q;
- the boundary domains $D_1^1, \dots, D_{\nu_k^1}^1$ adjacent to ∂Q.

Using the fact that the eigenfunctions, restricted to one of the four sides of Q, are trigonometric polynomials of degree at most $\sqrt{\mu_k(Q)}$, we get $\nu_k^1 \leq C\sqrt{\mu_k(Q)}$ for some constant C, so that $\lim_{k\to+\infty} \frac{\nu_k^1}{k} = 0$.
On the other hand, we can bound the interior domains as in the Dirichlet case. □

Neumann boundary condition: generalization

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a piecewise analytic boundary. We again consider the Laplacian in Ω with Neumann boundary condition, whose eigenvalues we denote by $(\mu_k)_{k \geq 1}$.

Theorem (J.A. Toth, S. Zelditch [2009])

For $k \geq 1$, we denote by r_k the greatest possible number of zeros of u on $\partial\Omega$, where u is an eigenfunction associated with μ_k. There exists a constant C_Ω such that

$$r_k \leq C_\Omega \sqrt{\mu_k}.$$

Theorem (I. Polterovich [2009])

Under the above hypotheses for Ω, for the Neumann-Laplacian eigenfunctions,

$$\limsup_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{4}{j^2}.$$

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Statement of the result

Let Ω be an open, bounded and connected open set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^{1,1}$.
For $h \in \text{Lip}\left(\overline{\Omega}\right)$ such that $h \geq 0$ on $\partial\Omega$, we consider the eigenvalue problem with Robin boundary condition

$$\left\{ \begin{array}{lcll} -\Delta u & = & \mu u & \text{in } \Omega; \\ \frac{\partial u}{\partial n} + h u & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

We denote by $(\mu_k)_{k\geq 1}$ the associated sequence of eigenvalues, arranged in innon-decreasing order and counted with multiplicities.
Using standard regularity result for elliptic boundary value problems, we can show that any eigenfunction of the above problem is of class $C^1\left(\overline{\Omega}\right)$.
As before, we denote by ν_k the maximal number of nodal domain for an eigenfunction associated with μ_k.

Theorem

We have the asymptotic upper bound

$$\limsup_{k\to+\infty} \frac{\nu_k}{k} \leq \gamma(n).$$

Preliminaries: inequalities in a nodal domain

Let (μ, u) is an eigenpair and D a nodal domain of u. We have

$$\int_D |\nabla u|^2 \leq \mu \int_D u^2.$$

Indeed,

$$\int_D |\nabla u|^2 = \int_D (-\Delta u)\, u + \int_{\partial D} \frac{\partial u}{\partial n} u = \mu \int_D u^2 - \int_{\partial D} h\, u^2 =$$
$$\mu \int_D u^2 - \int_{\partial D \cap \Omega} h\, u^2 - \int_{\partial D \cap \partial\Omega} h\, u^2 \leq \mu \int_D u^2.$$

Furthermore, the Faber-Krahn inequality gives a lower bound for the Rayleigh quotients. More precisely, for each $v \in H_0^1(D)$, we define

$$R(v, D) = \frac{\int_D |\nabla v|^2}{\int_D v^2},$$

and we have

$$\lambda_1 (\mathbb{B}^n)^{\frac{n}{2}}\, \omega_n \leq R(v, D)^{\frac{n}{2}} |D|.$$

The function v does not need to be a groundstate of $-\Delta$ in D.

Sorting the nodal domains I

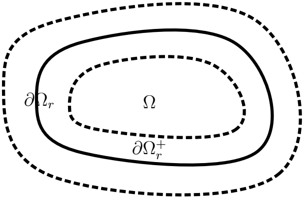
Given a nodal domain D, the main idea is to distinguish between two cases:

- most of the mass of the eigenfunction is inside Ω;
- there is a non-negligeable amount of mass near $\partial\Omega$.

This is a natural distinction when we try to apply the previous form of the Faber-Krahn inequality.

Given $r > 0$, we consider

$$\begin{aligned} \partial\Omega_r &:= \{x \in \mathbb{R}^n\,;\ \mathrm{dist}(x, \partial\Omega) < r\}; \\ \partial\Omega_r^+ &:= \partial\Omega_r \cap \Omega. \end{aligned}$$



For $\delta > 0$ small enough, we construct non-negative smooth functions φ_0 and φ_1 such that

- $\varphi_0^2 + \varphi_1^2 = 1$ in Ω,
- $\mathrm{supp}(\varphi_0) \subset \Omega \setminus \overline{\partial\Omega_{a\delta}^+}$ and $\mathrm{supp}(\varphi_1) \subset \partial\Omega_{A\delta}^+$,
- $\|\nabla\varphi_i\|_{L^\infty} \leq C\delta^{-1}$ for $i \in \{0, 1\}$,

with $0 < a < A$ and C independent of δ.

Sorting the nodal domains II

Let us consider an eigenpair (μ, u). We define $u_0 := \varphi_0 u$ and $u_1 := \varphi_1 u$.
By construction of (φ_0, φ_1), we have, for each nodal domain D of u,

$$\int_D u^2 = \int_D u_0^2 + \int_D u_1^2.$$

We fix $\varepsilon \in (0, 1)$. With respect to this choice, we say that D is

- a bulk domain if $\int_D u_0^2 \geq (1 - \varepsilon) \int_D u^2$;
- a boundary domain if $\int_D u_1^2 \geq \varepsilon \int_D u^2$.

We write

- $\nu^0(u, \varepsilon)$ for the number of bulk domains;
- $\nu^1(u, \varepsilon)$ for the number of boundary domains.

Ultimately, we want to show that $\nu^1(u, \varepsilon) \ll \nu^0(u, \varepsilon)$ for μ large. We therefore take δ depending on μ, namely

$$\delta := \mu^{-\theta}$$

with $\theta > 0$ to be determined.

Bulk domains

Let D be a bulk domain. We have $\lambda_1 (\mathbb{B}^n)^{\frac{n}{2}} \omega_n \leq R(u_0, D)^{\frac{n}{2}} |D|$.
On the one hand, by Young's inequality,

$$|\nabla u_0|^2 = |\varphi_0 \nabla u + u \nabla \varphi_0|^2 \leq (1+\varepsilon)\varphi_0^2 |\nabla u|^2 + \left(1 + \frac{1}{\varepsilon}\right) |\nabla \varphi_0|^2 u^2,$$

so that

$$\begin{aligned}\int_D |\nabla u_0|^2 \leq (1+\varepsilon) \int_D |\nabla u|^2 + \left(1 + \frac{1}{\varepsilon}\right) \frac{C}{\delta^2} \int_D u^2 \\ \leq \left((1+\varepsilon)\mu + \left(1 + \frac{1}{\varepsilon}\right) C^2 \mu^{2\theta}\right) \int_D u^2.\end{aligned}$$

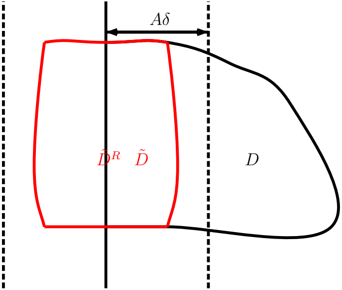
On the other hand $\int_D u_0^2 \geq (1-\varepsilon) \int_D u^2$. We get

$$R(u_0, D) \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\mu + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon} C^2 \mu^{2\theta}\right).$$

Summing the first inequality over all bulk domains, we obtain

$$\nu^0(u, \varepsilon) \leq \frac{|\Omega|}{\lambda_1 (\mathbb{B}^n)^{\frac{n}{2}} \omega_n} \left(\frac{1+\varepsilon}{1-\varepsilon}\mu + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon} C^2 \mu^{2\theta}\right)^{\frac{n}{2}}.$$

Boundary domains



We define

$$\widetilde{D} := D \cap \{u_1 \neq 0\} \subset \partial\Omega^+_{A\delta},$$

and u_1^R, $\widetilde{D}^R$ the reflection of u_1, $\widetilde{D}$ through $\partial\Omega$. We have in particular $\widetilde{D}^R \subset \partial\Omega_{A\delta}$.

By a similar computation as before, we get

$$R\left(u_1^R, \widetilde{D}^R\right) = R\left(u_1, \widetilde{D}\right) = \frac{\int_D |\nabla u_1|^2}{\int_D u_1^2} \leq \frac{2}{\varepsilon}\left(\mu + C^2\mu^{2\theta}\right).$$

On the other hand, Faber-Krahn inequality applied to $\widetilde{D}^R$ gives us

$$\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n \leq \left|R\left(u_1^R, \widetilde{D}^R\right)\right|^{\frac{n}{2}}\left|\widetilde{D}^R\right| = 2\left|R\left(u_1, \widetilde{D}\right)\right|^{\frac{n}{2}}\left|\widetilde{D}\right|$$

Summing over all boundary domains, we get

$$\nu^1(u,\varepsilon) \leq C' \frac{|\partial\Omega_{A\delta}|}{\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\omega_n}\left(\mu + C^2\mu^{2\theta}\right)^{\frac{n}{2}},$$

and therefore

$$\nu^1(u,\varepsilon) \leq C''\mu^{-\theta}\left(\mu + C^2\mu^{2\theta}\right)^{\frac{n}{2}}.$$

Conclusion of the proof

Let $(u_k)_{k\geq 1}$ be a sequence of eignefunction associated with (μ_k) and having each the maximal number of nodal domain, ν_k.
Let us recall that we have

$$\nu^0(u_k,\varepsilon) \leq \frac{|\Omega|}{\lambda_1\left(\mathbb{B}^n\right)^{\frac{n}{2}}\,\omega_n}\left(\frac{1+\varepsilon}{1-\varepsilon}\mu_k + \frac{1+\frac{1}{\varepsilon}}{1-\varepsilon}C^2\mu_k^{2\theta}\right)^{\frac{n}{2}}.$$

and

$$\nu^1(u_k,\varepsilon) \leq C''\mu^{-\theta}\left(\mu_k + C^2\mu_k^{2\theta}\right)^{\frac{n}{2}}.$$

We choose $\theta \in (0,\frac{1}{2})$, for instance $\theta = \frac{1}{4}$. Using Weyl's law, we have

$$\mu_k \leq \lambda_k \sim \frac{4\pi^2}{(\omega_n|\Omega|)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

$$\text{Therefore}\ \lim_{k\to+\infty}\frac{\nu^1(u_k,\varepsilon)}{k} = 0 \text{ and } \limsup_{k\to+\infty}\frac{\nu^0(u_k,\varepsilon)}{k} \leq \frac{(2\,\pi)^n}{\lambda\left(\mathbb{B}^n\right)^{\frac{n}{2}}\,\omega_n^2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n}{2}}.$$

Since $\nu_k = \nu^0(u_k,\varepsilon) + \nu^1(u_k,\varepsilon)$, we get

$$\limsup_{k\to+\infty}\frac{\nu_k}{k} \leq \frac{(2\,\pi)^n}{\lambda\left(\mathbb{B}^n\right)^{\frac{n}{2}}\,\omega_n^2}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n}{2}},$$

and the conclusion when $\varepsilon \to 0$.

Additional technical remarks

To prove the upper bound for $\nu^1(u,\varepsilon)$, we can introduce a (fixed) partition of unity $(\chi_i)_{1\leq i\leq N}$ covering $\partial\Omega$, and define local coordinates straightening $\partial\Omega$ in each $\text{supp}(\chi_i)$. This changes the constant but not the order of the upper bound.
The integrations by parts, reflections, ..., require some regularity of the nodal domains. To obtain it, we can work with super level sets insteads of nodal domains (following B. Bérard and D. Meyer [1982]). More precisely, if $u>0$ in the nodal domain D, we define

$$D_a := \{x \in D\,;\, u(x) > a\}$$

for $a>0$ small enough.
If a is a regular value for u, ∂D_a is a regular submanifold of $\mathbb{R}^n$ and we can make sense of the formal computations. According to Sard's theorem, we can find a sequence of positive regular values converging to 0. We then get the desired results by passing to the limit.

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Pleijel's argument as a minimal partition problem

Let Ω be an open, bounded and connected set in $\mathbb{R}^2$.
We call spectral k-equipartition (or k-partition for short) of Ω a family $\mathcal{D} = (D_1, \dots, D_k)$ of mutually disjoint, open and connected subsets of Ω satisfying $\lambda_1(D_1) = \cdots = \lambda_1(D_k)$.
We denote this common value by $\Lambda_k(\mathcal{D})$ and call it the energy of the partition.
For any $k \geq 1$, let us write

$$\mathfrak{L}_k(\Omega) = \inf\{\Lambda_k(\mathcal{D})\,;\, \mathcal{D} \text{ spectral } k\text{-equipartition of } \Omega\}.$$

Pleijel's proof establishes that

$$\frac{\mathfrak{L}_k}{k} \geq \frac{\lambda_1(\mathbb{D})|\mathbb{D}|}{|\Omega|} = \frac{\pi\, j^2}{|\Omega|}.$$

If we manage to prove that

$$\liminf_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq (1+\varepsilon)\frac{\pi j^2}{|\Omega|},$$

for some $\varepsilon > 0$, we improve the asymptotic upper bound to

$$\liminf_{k \to +\infty} \frac{\nu_k}{k} \leq \frac{1}{1+\varepsilon}\frac{4}{j^2}.$$

Quantitative improvements of Pleijel's theorem

Hexagonal Conjecture (M. van den Berg, L. Caffarelli, F. H. Lin, B. Helffer, T. Hoffmann-Ostenhof, S. Terracini, ...)

For large k, the spectral k-equipartition of Ω with minimal energy is close to a tilling of Ω by regular hexagons. In particular,

$$\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq \frac{\lambda_1(\mathrm{H})|\mathrm{H}|}{|\Omega|} \simeq \frac{18.5901}{|\Omega|}.$$

Theorem (J. Bourgain [2013] and S. Steinerberger [2013])

There exists $\varepsilon_0 > 0$ such that, for any domain Ω,

$$\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq (1+\varepsilon_0)\frac{\pi j^2}{|\Omega|}.$$

The argument by J. Bourgain is explicit and gives $\varepsilon_0 \simeq 4.10^{-9}$.
We know that

$$\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \frac{\lambda_1(\mathrm{H})|\mathrm{H}|}{|\Omega|},$$

which gives a limit on how far we can push Pleijel's argument (as pointed out by Steinerberger).

Conjecture on the optimal upper bound

Definition (B. Helffer, T. Hoffmann-Ostenhof [2015])

For a given domain $\Omega \subset \mathbb{R}^2$, we define Pleijel's constant by

$$Pl(\Omega) := \limsup_{k \to +\infty} \frac{\nu_k}{k}.$$

We have seen that for any domain $\Omega \subset \mathbb{R}^2$, $Pl(\Omega) < \gamma(2) = \frac{4}{j^2}$.

Conjecture (I. Polterovich [2009])

For any domain $\Omega \subset \mathbb{R}^2$, $Pl(\Omega) \leq \frac{2}{\pi}$.

Summary:

- Pleijel's theorem: $Pl(\Omega) \leq \frac{4}{j^2} \simeq 0.6916$;
- Hexagonal conjecture: $Pl(\Omega) \leq \frac{4\pi}{\lambda_1(\mathrm{H})|\mathrm{H}|} \simeq 0.6759$;
- Polterovich's conjecture: $Pl(\Omega) \leq \frac{4\pi}{\lambda_1(Q)|Q|} = \frac{2}{\pi} \simeq 0.6366$.

Let us consider the rectangle $\mathrm{R}(a,b) := (0, \pi\, a) \times (0, \pi\, b)$.

Proposition (B. Helffer, T. Hoffmann-Ostenhof [2015])

If $(b/a)^2 \notin \mathbb{Q}$, we have $Pl(\mathrm{R}(a,b)) = 2/\pi$.

Proof for irrational rectangles

The eigenvalues of the Dirichlet-Laplacian in $\mathrm{R}(a,b)$ are

$$\lambda_{m,n} = \frac{m^2}{a^2} + \frac{n^2}{b^2},$$

and a basis of eigenfunctions is given by

$$u_{m,n}(x,y) = \cos\left(\frac{m\,x}{a}\right)\,\cos\left(\frac{n\,y}{b}\right).$$

We recall that the counting function N is defined by

$$N(\lambda) = \sharp\{\ell \in \mathbb{N}^*\,;\,\lambda_\ell(\mathrm{R}(a,b)) \leq \lambda\}.$$

Since the eigenvalues are simple, $N\left(\lambda_k(\mathrm{R}(a,b))\right) = k$. For $k \in \mathbb{N}^*$, we define the pair (m_k, n_k) by $\lambda_{m_k,n_k} = \lambda_k(\mathrm{R}(a,b))$. We therefore have, as $k \to +\infty$,

$$\frac{\nu_k}{k} = \frac{m_k\,n_k}{N\left(\lambda_k(\mathrm{R}(a,b))\right)} \sim \frac{m_k\,n_k}{\frac{\pi^2\,a\,b}{4\,\pi}\left(\left(\frac{m_k}{a}\right)^2 + \left(\frac{n_k}{b}\right)^2\right)} = \frac{4}{\pi}\,\frac{\frac{a\,n_k}{b\,m_k}}{1 + \left(\frac{a\,n_k}{b\,m_k}\right)^2}.$$

By a suitable choice of a sequence of pairs (m,n), we obtain that the set of limits points of the sequence $\left(\frac{\nu_k}{k}\right)_{k\geq 1}$ is the segment $[0, 2/\pi]$.

Open problems

- Is it true that, for all domains, there exists a sequence of eigenfunctions, associated with increasing eigenvalues, having an increasing number of nodal domains?
- Do we have
$$\limsup_{k\to+\infty} \frac{\nu_k}{k} > 0$$
for any domains?
- Does
$$\liminf_{k\to+\infty} \frac{\nu_k}{k} = 0$$
for any domain?
- Is Pleijel's constant the same for all boundary conditions?
- Can we get explicit bounds of the number of eigenvalues satisfying the case of equality in Courant theorem? Some are know for the Dirichlet boundary conditions. What about Neumann and Robin?
- How to use the fact that we consider nodal partition and not mainly spectral equipartition?
- Does it help to consider averaged/probabilistic versions?