Spectral estimates for punctured domains and Aharonov-Bhom operators
Part 2: Singular perturbation of Dirichlet-Laplacian eigenvalues

Joint work with Laura Abatangelo, Veronica Felli (Milano-Bicocca) and Luc Hillairet (Orléans)

Corentin Léna

Università degli Studi di Torino

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Setting

Let Ω be an open, bounded, and connected set in $\mathbb{R}^n$.

We define the eigenvalues $(\lambda_j(\Omega))_{j\geq 1}$ of the Dirichlet Laplacian (counted with multiplicies) by

$$\begin{cases} -\Delta u_j &= \lambda_j(\Omega)\, u_j \text{ in } \Omega, \\ u_j &\in H^1_0(\Omega) \end{cases}$$

for some normalized eigefunction u_j: $\int_\Omega |u_j|^2\, dx = 1$.

To avoid any regularity assumption for Ω, they can be defined using the min-max formula.

General question: how do the eigenvalues change when the domain Ω is varied?

Regular perturbation theory (Hadamard's formula): if $\partial\Omega$ is of class C^2, V is a bounded vector field fo class C^1 , $\Omega_t := (Id + tV)(\Omega)$ for small t, and $\lambda_k(\Omega)$ is a simple eigenvalue then

$$\frac{d}{dt}\lambda_k(\Omega_t)\bigg|_{t=0} = -\int_{\partial\Omega}\left(\frac{\partial u_k}{\partial \nu}\right)^2 \nu \cdot V.$$

Singular perturbations: examples

The Crushed Ice Problem (J. Rauch and M. Taylor, 1974):

Let us define the sequence of domains $(\Omega_p)_{p\geq 1}$ such that Ω_p is $\Omega \subset \mathbb{R}^3$ (open, connected, and bounded) with p balls of radius $r_p > 0$ removed. Let us assume that $p\, r_p \to 0$. Then $\lambda_j(\Omega_p) \to \lambda_j(\Omega)$ for all $j \geq 1$.

They actually treat $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) and prove a stronger result (convergence of the resolvent). However, no information on the rate of convergence.

Domain with a small hole (S. Ozawa,1981):

Let us define $\Omega_\varepsilon = \Omega \setminus \overline{B}(x_0, \varepsilon)$ for x_0 a fixed point in $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$ small enough. Assume that $\lambda_k(\Omega)$ is a simple eigenvalue, and that u_k is an associated normalized eigenfunction. Then, if $n = 2$,

$$\lambda_k(\Omega_\varepsilon) = \lambda_N(\Omega) + \frac{2\pi}{|\log(\varepsilon)|} u_N(x_0)^2 + O\left(\frac{1}{\log(\varepsilon)^2}\right),$$

and, if $n = 3$,

$$\lambda_N(\Omega_\varepsilon) = \lambda_N(\Omega) + 4\pi\varepsilon u_N(x_0)^2 + O\left(\varepsilon^{\frac{3}{2}}\right).$$

The proof uses an asymptotic expansion of the Green function. If $u_N(x_0) = 0$, we don't get the leading order term.

Classical capacity

If $K \subset \Omega \subset \mathbb{R}^n$ is a compact set, we fix $\eta_K \in C_c^\infty(\Omega)$ with $\eta_K \equiv 1$ in a neighborhood of K. To characterize small sets for the perturbation of Dirichlet eigenvalues, we define

$$\mathrm{Cap}_\Omega(K) = \inf \left\{ \int_\Omega |\nabla f|^2 \, dx : \; f \in H_0^1(\Omega) \text{ and } f - \eta_K \in H_0^1(\Omega \setminus K) \right\}.$$

The infimum is reached by the capacitary potential $V_K \in H_0^1(\Omega)$ so that

$$\mathrm{Cap}_\Omega(K) = \int_\Omega |\nabla V_K|^2 \; dx.$$

If K is regular (closure of an open and smooth set), V_K is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta V_K = 0, & \text{in } \Omega \setminus K, \\ V_K = 0, & \text{on } \partial\Omega, \\ V_K = 1, & \text{on } K. \end{cases}$$

Case of a manifold

If X is a compact Riemannian manifold without boundary, we consider the eigenvalues of the Laplace-Beltrami operator. For $K \subset X$ a compact set, we define

$$\mathrm{Cap}_X(K) = \inf \left\{ \int_X |\nabla f|^2 \, dx : \; f \in H^1(X), \; \int_X f \, dx = 0, \text{ and } f - 1 \in H^1_0(X \setminus K) \right\}.$$

Theorem (G. Courtois, 1995)

Let $\lambda := \lambda_k(X) = \ldots = \lambda_{k+m-1}(X)$ be a eigenvalue of $-\Delta_X$ of multiplicity m. There exists a function $r : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t \to 0} r(t) = 0$ and a positive constant ε_k such that, for any compact set K of X, if $\mathrm{Cap}_X(K) \leq \varepsilon_k$, then

$$|\lambda_{k+j}(X \setminus K) - \lambda_{k+j}(X) - \mathrm{Cap}_X(K) \cdot \mu_K(u_{k+j}^2)| \leq \mathrm{Cap}_X(K) \cdot r(\mathrm{Cap}_X(K))$$

for $0 \leq j \leq m-1$, where μ_K is a probability measure supported on K and $(u_k, \ldots, u_{k+m-1})$ is an orthonormal basis of the eigenspace of λ which diagonalizes the quadratic form $u \mapsto \mu_A(u^2)$ according to the increasing order of its eigenvalues.

From now on, we consider the planar case ($n = 2$).
Let $x_0 \in \Omega \subset \mathbb{R}^2$, $u \in H^1_0(\Omega)$ and $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact and connected sets contained in Ω. We write $\Omega_\varepsilon := \Omega \setminus K_\varepsilon$.
We say that $(K_\varepsilon)_\varepsilon$ concentrates to x_0 if

$$\max_{y \in K_\varepsilon} |y - x_0| \to 0 \text{ as } \varepsilon \to 0^+.$$

This implies in particular that $\operatorname{diam}(K_\varepsilon) \to 0$ as $\varepsilon \to 0^+$.
Let us assume that $\lambda_k(\Omega)$ is simple, and let u_k be a normalized associated eigenfunction.

Theorem

If $u_k(x_0) \neq 0$ and $(K_\varepsilon)_\varepsilon$ concentrates to x_0, then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \frac{2\pi}{|\log(\operatorname{diam}(K_\varepsilon))|} u_k(0)^2 \text{ as } \varepsilon \to 0^+.$$

This is essentially a particular case of the result by G. Courtois, in a slightly different situation, combined with an explicit estimate of the capacity.
This generalizes one of the results by Ozawa, since it allows holes of irregular shape, whose shape changes with ε.

We now consider the case where $u_k(x_0) = 0$. Since u_k is a Laplacian eigenfunction, there exist an integer $m \geq 1$, $\alpha \in [0, \pi)$ and $C \neq 0$ such that

$$u_k\left((x_0)_1 + r\cos\theta, (x_0)_2 + r\sin\theta\right) = C\, r^m \sin(\alpha - m\theta) + O\left(r^{m+1}\right).$$

Theorem

If $K_\varepsilon = B(x_0, \varepsilon)$, then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim 2\, m\, \pi\, C^2\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+.$$

In particular, let us assume that x_0 is a regular point in the nodal set of u_k, i.e. that $u_k(x_0) = 0$ and $\nabla u_k(x_0) \neq 0$. The theorem tells us that

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim 2\, \pi\, |\nabla u_k(x_0)|^2\, \varepsilon^2 \text{ as } \varepsilon \to 0^+.$$

Theorem

If K_ε is the segment $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$ and if $\alpha \neq 0$ (that is to say if the segment is not tangent to a nodal line at x_0) then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \pi\, C^2\, C_m \sin(\alpha)^2\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+,$$

where C_m is a positive constant depending only on m and known explicitly.

In particular, if x_0 is a regular point in the nodal set of u_k,

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \pi\, |\nabla u_k(x_0)|^2\, (\sin\alpha)^2 \varepsilon^2 \text{ as } \varepsilon \to 0^+.$$

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u-capacity

For any $u \in H_0^1(\Omega)$, we define

$$\mathrm{Cap}_\Omega(K,u) = \inf \left\{ \int_\Omega |\nabla f|^2 : \ f \in H_0^1(\Omega) \text{ and } f - u \in H_0^1(\Omega \setminus K) \right\}.$$

This can be extended to $u \in H^1(\Omega)$ by setting $\mathrm{Cap}_\Omega(K,u) := \mathrm{Cap}_\Omega(K, \eta_K u)$, where $\eta_K \in C_c^\infty(\Omega)$ with $\eta_K \equiv 1$ in a neighborhood of K. Then $\mathrm{Cap}_\Omega(K) = \mathrm{Cap}_\Omega(K,1)$.

This quantity was introduced by G. Courtois (1995) to obtain asymptotic expansions. It is called the Dirichlet capacity when $u = u_1$ and can be used to obtain spectral stability results (J. Bertrand and B. Colbois, 2005).

The minimizer is called the capacitary potential $V_{K,u}$. In the case where K is regular, $V_{K,u}$ solves the Dirichlet problem

$$\begin{cases} \Delta V_{K,u} = 0 & \text{in } \Omega \setminus K; \\ V_{K,u} = 0 & \text{on } \partial\Omega; \\ V_{K,u} = u & \text{on } K. \end{cases}$$

The following monotonicity properties follow from the definition:

- if $K_1 \subset K_2 \subset \Omega$, $\mathrm{Cap}_\Omega(K_1,u) \le \mathrm{Cap}_\Omega(K_2,u)$;
- if $K \subset \Omega_1 \subset \Omega_2$, $\mathrm{Cap}_{\Omega_2}(K,u) \le \mathrm{Cap}_{\Omega_1}(K,u)$.

Compact set concentrating to a point

We still assume that $(K_\varepsilon)_{\varepsilon>0}$ is a family of compact sets in Ω.

Proposition

Let $u \in H_0^1(\Omega)$. For $\varepsilon > 0$, $V_{K_\varepsilon,u}$ is the capacitary potential associated with $\mathrm{Cap}_\Omega(K_\varepsilon, u)$. If $(K_\varepsilon)_{\varepsilon>0}$ concentrates to x_0,

$$\|V_{K_\varepsilon,u}\| = o\left(\mathrm{Cap}_\Omega(K_\varepsilon, u)^{\frac{1}{2}}\right) \text{ as } \varepsilon \to 0^+.$$

Proof.

Assume by contradiction that this is not the case. Then there exist some constant C and a sequence $\varepsilon_n \to 0^+$ such that

$$\int_\Omega |\nabla V_{\varepsilon_n}|^2 \leq C \int_\Omega |V_{\varepsilon_n}|^2.$$

We set $W_n := V_{\varepsilon_n}/\|V_{\varepsilon_n}\|$. The sequence (W_n) is bounded in $H_0^1(\Omega)$. Up to subsequences, it converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to some $W_\infty \in H_0^1(\Omega)$. We can check that W_∞ is harmonic in $\Omega \setminus \{x_0\}$ and therefore in Ω. The maximum principle implies that $W_\infty = 0$, contradicting $\|W_\infty\| = 1$. □

Spectral estimate

Theorem

Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact sets in Ω concentrating to $x_0 \in \Omega$. Let us assume that $\lambda_k(\Omega)$ is simple, and let u_k be a normalized associated eigenvalue. We write $\Omega_\varepsilon := \Omega \setminus K_\varepsilon$. Then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \mathrm{Cap}_\Omega(K_\varepsilon, u_k) \text{ as } \varepsilon \to 0^+.$$

Let us prove the theorem. We write $\lambda := \lambda_k(\Omega)$, $u := u_k$, $\lambda(\varepsilon) := \lambda_k(\Omega_\varepsilon)$, $c(\varepsilon) := \mathrm{Cap}_\Omega(K_\varepsilon, u_k)$, $V_\varepsilon := V_{K_\varepsilon, u_k}$ and $v_\varepsilon := u - V_\varepsilon$.
We have

$$(-\Delta - \lambda)v_\varepsilon = (-\Delta - \lambda)(u - V_\varepsilon) = \lambda V_\varepsilon = o\left(c(\varepsilon)^{\frac{1}{2}}\right).$$

By continuity, $\lambda(\varepsilon)$ is simple for $\varepsilon > 0$ small enough. As a consequence of the spectral theorem,

$$|\lambda(\varepsilon) - \lambda| = o\left(c(\varepsilon)^{\frac{1}{2}}\right)$$

Let us denote by u_ε the orthogonal projection of u on the eigenspace $E(\lambda(\varepsilon))$.
We also have

$$\|u_\varepsilon - u\| = o\left(c(\varepsilon)^{\frac{1}{2}}\right).$$

We have

$$c(\varepsilon) = \int_\Omega |\nabla V_\varepsilon|^2 = \int_\Omega \nabla V_\varepsilon \cdot \nabla u = -\int_\Omega V_\varepsilon \Delta u = \lambda \langle V_\varepsilon, u \rangle.$$

On the one hand

$$\langle (-\Delta - \lambda) v_\varepsilon, u_\varepsilon \rangle = \langle v_\varepsilon, (-\Delta - \lambda) u_\varepsilon \rangle = (\lambda(\varepsilon) - \lambda) \langle v_\varepsilon, u_\varepsilon \rangle.$$

On the other hand,

$$\langle (-\Delta - \lambda) v_\varepsilon, u_\varepsilon \rangle = \lambda \langle V_\varepsilon, u_\varepsilon \rangle = \lambda \langle V_\varepsilon, u \rangle + \lambda \langle V_\varepsilon, u_\varepsilon - u \rangle,$$

so that

$$\langle (-\Delta - \lambda) v_\varepsilon, u_\varepsilon \rangle = \lambda \langle V_\varepsilon, u \rangle + o\left(c(\varepsilon)\right).$$

Finally,

$$\lambda(\varepsilon) - \lambda = \frac{c(\varepsilon) + o\left(c(\varepsilon)\right)}{\langle v_\varepsilon, u_\varepsilon \rangle},$$

which gives the desired result.

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Theorem

Let $u \in H_0^1(\Omega) \cap C^2(\Omega)$ and let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact and connected sets concentrating to a point $x_0 \in \Omega$. Assume $u(x_0) \neq 0$. Then

$$\mathrm{Cap}_\Omega(K_\varepsilon, u) \sim \frac{2\pi}{|\log(\mathrm{diam}(K_\varepsilon))|} u(x_0)^2 \text{ as } \varepsilon \to 0^+.$$

Proposition

Under the hypotheses of the theorem

$$\mathrm{Cap}_\Omega(K_\varepsilon, u) \sim u(x_0)^2 \mathrm{Cap}_\Omega(K_\varepsilon) \text{ as } \varepsilon \to 0^+.$$

Proof.

Let us denote by V_u and V the potentials associated with $\mathrm{Cap}_\Omega(K_\varepsilon, u)$ and $\mathrm{Cap}_\Omega(K_\varepsilon)$ respectively. Assuming ∂K_ε regular enough, we find after some computation

$$\mathrm{Cap}_\Omega(K_\varepsilon, u) = \int_{\partial\Omega_\varepsilon} u^2 \partial_\nu V - \int_\Omega V V_u \Delta u - 2\int_\Omega V_u \nabla V \cdot \nabla u.$$

Furthermore, $\mathrm{Cap}_\Omega(K_\varepsilon) = \int_{\partial\Omega_\varepsilon} \partial_\nu V$, and $\partial_\nu V \geq 0$ according to Hopf's Lemma.

Proposition

If $(K_\varepsilon)_{\varepsilon>0}$ is a family of compact and connected sets concentrating to $x_0 \in \Omega$, then

$$\mathrm{Cap}_\Omega (K_\varepsilon) \sim \frac{2\pi}{|\log(\mathrm{diam}(K_\varepsilon))|} \text{ as } \varepsilon \to 0^+.$$

To obtain an upper bound, let us chose $x_\varepsilon \in K_\varepsilon$ for each $\varepsilon > 0$, and let us fix $R > 0$ such that $B(x_\varepsilon, R) \subset \Omega$ for all $\varepsilon > 0$ small enough. Let us write $d_\varepsilon := \mathrm{diam}(K_\varepsilon)$.
We have $K_\varepsilon \subset B(x_\varepsilon, d_\varepsilon) \subset B(x_\varepsilon, R) \subset \Omega$. According the the monotonicity properties for the capacity,

$$\mathrm{Cap}_\Omega (K_\varepsilon) \le \mathrm{Cap}_{B(x_\varepsilon,R)} (B(x_\varepsilon, d_\varepsilon)) = \mathrm{Cap}_{B(0,R)} (B(0, d_\varepsilon)).$$

The potential for the capacity on the right is

$$V(x) = \frac{\log\left(\frac{|x|}{R}\right)}{\log\left(\frac{d_\varepsilon}{R}\right)}$$

and therefore

$$\mathrm{Cap}_{B(0,R)} (B(0, d_\varepsilon)) = \frac{2\pi}{\log\left(\frac{d_\varepsilon}{R}\right)} \sim \frac{2\pi}{|\log(d_\varepsilon)|} \text{ as } \varepsilon \to 0^+.$$

The lower bound is harder to obtain. Let us begin by computing the capacity of a segment in an ellipse.
For $D, \delta > 0$, we define the interior of the ellipse of semi-axes $\sqrt{D^2+\delta^2}$ and D:

$$\mathcal{E}(D,\delta) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \, ; \, \frac{x_1^2}{D^2+\delta^2} + \frac{x_2^2}{L^2} < 1 \right\}.$$

Let us note $s_\delta := [-\delta, \delta] \times \{0\}$. The elliptic coordinates

$$\begin{cases} x_1 &= \delta \cosh(\xi) \cos(\eta); \\ x_2 &= \delta \sinh(\xi) \sin(\eta); \end{cases}$$

map conformaly the rectangle $(0, \xi(D,\delta)) \times [0, 2\pi)$ to $\mathcal{E}(D,\delta) \setminus s_\delta$, where

$$\xi(D,\delta) := \operatorname{argsinh}\left(\frac{\delta}{D}\right).$$

In the elliptic coordinates, the potential is

$$V(\xi,\eta) = 1 - \frac{\xi}{\xi(D,\delta)},$$

and therefore we obtain

$$\mathrm{Cap}_{\mathcal{E}(D,\delta)}(s_\varepsilon) = \frac{2\pi}{\xi(D,\delta)} = 2\pi \operatorname{argsinh}\left(\frac{\delta}{D}\right)^{-1} \sim \frac{2\pi}{|\log(\delta)|} \text{ as } \delta \to 0^+ \text{ with } D \text{ fixed.}$$

For $\varepsilon > 0$, there exist x_ε and y_ε such that $|y_\varepsilon - x_\varepsilon| = \text{diam}\,(K_\varepsilon)$. Let us note $s_\varepsilon = [x_\varepsilon, y_\varepsilon]$, $\delta_\varepsilon = \frac{1}{2}\text{diam}\,(K_\varepsilon)$ and L_ε the line containing the segment s_ε.
Let us fix $D > 0$ large enough so that, for all $\varepsilon > 0$, $\Omega \subset \mathcal{E}_\varepsilon$, where $\mathcal{E}_\varepsilon$ is the ellipse of foci x_ε and y_ε, major semi-axis $\sqrt{D^2 + \delta_\varepsilon^2}$ and minor semi-axis D. By monotonicity of the capacity,

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon) \leq \mathrm{Cap}_\Omega(K_\varepsilon).$$

On the other hand,

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon}(s_\varepsilon) \leq \mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon).$$

Indeed, let us denote by K_ε^* the Steiner symmetrization of K_ε with respect to the line L_ε. It leaves $\mathcal{E}_\varepsilon$ invariant, $s_\varepsilon \subset K_\varepsilon^*$ by connectedness of K_ε, and

$$\mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon^*) \leq \mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon).$$

The claim follows by monotonicity.
We finally obtain

$$\frac{2\pi}{|\log(\delta_\varepsilon)|}(1 + o(1))) \leq \mathrm{Cap}_\Omega(K_\varepsilon),$$

which gives the correct lower bound.

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To simplify notation, let us assume that $x_0 = 0$.
Let $m \geq 1$ be an integer. If $u \in C^{m+1}(\Omega)$ and 0 is a zero of order m of u, then

$$u(x) = P_m(x) + h(x),$$

where P_m is a homogeneous polynomial of degree m, $h \in C^{m+1}(\Omega)$, $h(x) = O\left(|x|^{m+1}\right)$, and $|\nabla h(x)| = O\left(|x|^m\right)$ as $|x| \to 0$.

Lemma

Let h be a function satisfying the above hypotheses and let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact sets such that there exists $\varepsilon_0 > 0$ and C such that $\max_{y \in K_\varepsilon} |y| \leq C\varepsilon$ for $\varepsilon \in (0, \varepsilon_0)$. Then

$$\mathrm{Cap}_\Omega\left(K_\varepsilon, h\right) = O\left(\varepsilon^{2n+2}\right).$$

Proposition

Let u and $(K_\varepsilon)_{\varepsilon>0}$ satisfy the hypotheses above, and let us assume that there existe a positive constant α such that

$$\mathrm{Cap}_\Omega\left(K_\varepsilon, P_m\right) \sim \alpha\varepsilon^{2m} \text{ as } \varepsilon \to 0^+.$$

Then

$$\mathrm{Cap}_\Omega\left(K_\varepsilon, u\right) \sim \alpha\varepsilon^{2m} \text{ as } \varepsilon \to 0^+.$$

Small disks

We set $K_\varepsilon = B_\varepsilon = B(0,\varepsilon)$. We choose $0 < R_1 < R_2$ such that $B(0,R_1) \subset \Omega \subset B(0,R_2)$ for $\varepsilon > 0$ small enough. Then

$$\mathrm{Cap}_{B(0,R_2)}(B_\varepsilon, P_m) \leq \mathrm{Cap}_{\Omega}(B_\varepsilon, P_m) \leq \mathrm{Cap}_{B(0,R_1)}(B_\varepsilon, P_m).$$

To compute $\mathrm{Cap}_{B(0,R)}(B_\varepsilon, P_m)$ for $R > \varepsilon > 0$, we write

$$P_m(\cos\theta, \sin\theta) = \frac{a_0}{2} + \sum_{j=1}^{m} (a_j \cos j\theta + b_j \sin j\theta).$$

We solve a Dirichlet problem for each term of the sum and find

$$\mathrm{Cap}_{B(0,R)}(B_\varepsilon, P_m) \sim \pi\, D(P_m)\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+, \ R \text{ fixed},$$

where

$$D(P_m) = \frac{m\, a_0^2}{4} + \sum_{j=1}^{m} \frac{(m+j)^2}{2m} \left(a_j^2 + b_j^2\right)^2.$$

If u is a Laplacian eigenfunction, the homogeneous polynomial P_m is harmonic, so that there exist $C \neq 0$ and $\alpha \in [0,\pi)$ such that

$$P_m(r\cos\theta, r\sin\theta) = r^m \sin(\alpha - m\theta).$$

Therefore $a_0 = a_1 = b_1 = \cdots = a_{m-1} = b_{m-1} = 0$, $a_m^2 + b_m^2 = C^2$ and $D(P_m) = 2m\, C^2$.

Small segments

We set $K_\varepsilon = s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. We chose $0 < L_1 < L_2$ such that $\mathcal{E}(\varepsilon, L_1) \subset \Omega \subset \mathcal{E}(\varepsilon, L_2)$ for $\varepsilon > 0$ small enough. Then

$$\mathrm{Cap}_{\mathcal{E}(\varepsilon,L_2)}(s_\varepsilon, P_m) \le \mathrm{Cap}_{\Omega}(s_\varepsilon, P_m) \le \mathrm{Cap}_{\mathcal{E}(\varepsilon,L_1)}(s_\varepsilon, P_m).$$

To compute $\mathrm{Cap}_{\mathcal{E}(\varepsilon,L)}(s_\varepsilon, P_m)$ for L, $\varepsilon > 0$, we write
$P_m(x_1, x_2) = \sum_{j=0}^m c_j\, x_1^{m-j} x_2^j$.
In elliptic coordinates, for $\xi = 0$, $P_m(\varepsilon \cos\eta, 0) = c_0\, \varepsilon^m (\cos\eta)^m$. We write

$$(\cos\eta)^m = \sum_{j=0}^{m} A_j \cos j\eta.$$

By solving the Dirichlet problem in elliptic coordinates for each term, we find

$$\mathrm{Cap}_{\mathcal{E}(\varepsilon,L)}(s_\varepsilon, P_m) \sim \pi\, C_m\, c_0^2\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+,$$

with

$$C_m = \sum_{j=1}^{m} j\, A_j^2.$$

If $P_m(\cos\theta, \sin\theta) = C \sin(\alpha - m\theta)$, then $c_0 = C\sin(\alpha)$, and therefore

$$\mathrm{Cap}_{\mathcal{E}(\varepsilon,L)}(s_\varepsilon, P_m) \sim \pi\, C_m\, C^2 (\sin\alpha)^2 \varepsilon^{2m} \text{ as } \varepsilon \to 0^+.$$

Open problems

- Can we remove the symmetry hypothesis for the Aharonov-Bohm problem? Can we find a capacity adapted to the Aharonov-Bohm operator?
- Can we treat multiple eigenvalues (in both problems)? We should show convergence to a finite dimensional problem, but what should it look like?
- Can we go to a higher order? In the punctured domain problem, when we make the hole on the nodal set, we probably do not have an expansion in power of the capacity. Where does it break down?