Spectral estimates for punctured domains and Aharonov-Bhom operators
Part 1: Eigenvalues of Aharonov-Bohm operators

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Introduction

Spectral properties of Aharonov-Bohm operators

Collapse of two poles away from the nodal set

Collapse of two poles on the nodal set

A result from superconductivity

Let us begin with results obtained by J. Berger and J. Rubinstein (1999) for 2-dimensional superconductivity.
They consider the Ginzburg-Landau model: the state of a superconducting material in $\Omega \subset \mathbb{R}^2$ is described by minimizer $(u, \mathbf{A})$ of the Ginzburg-Landau functional. The vector field $\mathbf{A}$ is a magnetic potential: $B := \partial_1 A_2 - \partial_2 A_1$ while $|u|$ gives the density of superconducting electrons.
In general, we expect the zeroset of u to consists at most of isolated points.
They assume the domain Ω is ring-like (figure) and submitted to an applied magnetic field B_e. The quantity of interest is the reduced flux of B_e:

$$\Phi := \frac{1}{2\pi} \int_{\omega} B_e.$$

Theorem

If $\Phi \in \mathbb{Z} + \frac{1}{2}$, B_e vanishes in $\overline{\Omega}$ and Ω satisfies an additional simplicity assumption, there is a range of temperatures below the critical value such that u vanishes on a curve connecting the inner with the outer boundary.

By a bifurcation analysis, they reduce this to the study of the following linear eigenvalue problem.

$$(P)\begin{cases} (i\nabla + \mathbf{A}_e)^2\, u & = \quad \lambda\, u \text{ in } \Omega; \\ (i\nabla + \mathbf{A}_e)\, u \cdot \nu & = \quad 0 \text{ on } \partial\Omega. \end{cases}$$

Theorem

Assume $\Phi \in \mathbb{Z} + \frac{1}{2}$, B_e vanishes in $\overline{\Omega}$ and λ_1, the lowest eigenvalue of problem (P), is simple. Let u_1 be an associated eigenfunction. The zeroset of u_1 consists of a curve connecting the inner with the outer boundary.

They point out another quantum mechanical interpretation of (P): the Schrödinger equation for a charged particle in Ω submitted to B_e, with a zero probability current at the boundary (verify) . Then, there is a forbidden curve in the groundstate. This is a manifestation of the Aharonov-Bohm effect, since B_e vanishes in $\overline{\Omega}$.

A spectral minimal partition problem

Given a bounded open set $\omega \subset \mathbb{R}^2$, we will in the following denote by $(\lambda_j(\omega))_{j\geq 1}$ the eigenvalues of the Dirichlet realization of the Laplacian in ω, arranged in non-decreasing order and counted with multiplicities.
Given $\Omega \subset \mathbb{R}^2$ open, bounded and connected, with a regular enough boundary, a k-partition $\mathcal{D}$ of Ω is a family of k mutually disjoint connected open sets in Ω

$$\mathcal{D} = (D_1, \ldots, D_k).$$

We consider the minimization problem

$$(M) \quad \min_{\mathcal{D}} \max_{i\in\{1,\ldots,k\}} \lambda_1(D_i).$$

Existence and regularity of solution for Problem (M) follow from the work of several authors [Bucur, Buttazzo, and Henrot (1998); Conti, Terracini, and Verzini (2005); Caffarelli and Lin (2007); Helffer, Hoffmann-Ostenhof, and Terracini (2009)].
By regularity, we mean that

- $\overline{\Omega} = \bigcup_{i=1}^{k} \overline{D}_i$;
- $N(\mathcal{D}) := \Omega \setminus \bigcup_{i=1}^{k} D_i$ consists of regular arcs meeting with equal angles at a finite number of multiple points, in Ω or on $\partial\Omega$.

Furthermore, if $\mathcal{D} = (D_1, \ldots, D_k)$ is minimal, it is a spectral equipartition:

$$\lambda_1(D_1) = \cdots = \lambda_1(D_i) = \cdots = \lambda_1(D_k).$$

Nodal partitions

Given a regular spectral equipartition $\mathcal{D} = (D_1, \ldots, D_k)$, we say that the domains D_i and D_j are neighbors if Int $\left(\overline{D_i \cup D_j}\right)$ is connected. We say that $\mathcal{D}$ is bipartite if we can color its domains in such a way that neighbors have different colors, using only two colors.

Theorem

A minimal partition is nodal (i.e. consists of the nodal domain of an eigenfunction of $-\Delta_\Omega^D$)) if, and only if, it is bipartite.

Proposition

If u is an eigenfunction associated with $\lambda_k(\Omega)$ with k nodal domains, the corresponding nodal partition is minimal.

Proof.

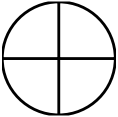
Let $\mathcal{D} = (D_1, \ldots, D_k)$ be a spectral equipartition of energy Λ and let $\Omega' := \bigcup_{i=1}^k D_i$. We have $\Lambda = \lambda_k(\Omega') \geq \lambda_k(\Omega)$, and $\lambda_k(\Omega)$ is the energy of the nodal k-partition associated with u. □

Theorem

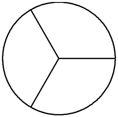
If a minimal k-partition is nodal, it is associated with $\lambda_k(\Omega)$.

A 3-partition of the disk

In the case of the unit disk $\mathbb{D}$, $\lambda_1(\mathbb{D}) < \lambda_2(\mathbb{D}) = \lambda_3(\mathbb{D}) < \lambda_4(\mathbb{D})$.
The partition of $\mathbb{D}$ into four equal sectors is minimal.



A minimal 3-partitions of $\mathbb{D}$ are not nodal.
The following partition $\mathcal{D}^*$ of the unit disk $\mathbb{D}$ is conjectured to be minimal.



The partition $\mathcal{D}^*$ is not bipartite and therefore not nodal.
Let us define $\Omega' := \mathbb{D} \setminus [0, 1]$. Then $\mathcal{D}^*$ is a nodal partition of Ω', associated with the eigenvalue $\lambda_3(\Omega') = j^2_{\frac{3}{2},1}$, and given by the Courant-sharp eigenfunction

$$u(r, \theta) := J_{\frac{3}{2}}\left(j_{\frac{3}{2},1} r\right) \sin\left(\frac{3\theta}{2}\right).$$

Construction of a magnetic Schrödinger operator

Let us define

$$u^*(r,\theta) = e^{i\frac{\theta}{2}} u(r,\theta).$$

Then u^* admits a smooth (complex valued) extension in the punctured disk $\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\}$.

The function u^* satisfies the eigenvalue equation

$$\left(i\nabla + \frac{1}{2}\mathbf{A}_1^0 \right)^2 u^* = \lambda_3(\Omega')u^*,$$

with

$$\mathbf{A}_1^0(r,\theta) := \frac{1}{r}\mathbf{e}_\theta .$$

More generally, we can define a self-adjoint realization of the differential operator $(i\nabla + {}^1\!/\!{}_2\,\mathbf{A}_1^0)^2$, which we denote by $H^0_{1/2}$. Its spectrum consists of positive eigenvalues with finite multiplicities. We denote it by $(\lambda_j(0,1/2))_{j\geq 1}$ (counting with multiplicities). The construction could be done for any domain $\Omega \ni 0$.

In this particular example, the eigenvalues are $\left\{ j^2_{n+\frac{1}{2},\ell};\, n \in \mathbb{N} \text{ and } \ell \in \mathbb{N}^* \right\}$, each with multiplicity 2. An associated basis of eigenfunction is given by

$$u^c_{n,\ell}(r,\theta) = J_{n+\frac{1}{2}}\left(j_{n+\frac{1}{2},\ell}\, r\right) e^{i\frac{\theta}{2}} \cos\left(\left(n+\frac{1}{2}\right)\theta\right)$$

and

$$u^c_{n,\ell}(r,\theta) = J_{n+\frac{1}{2}}\left(j_{n+\frac{1}{2},\ell}\, r\right) e^{i\frac{\theta}{2}} \sin\left(\left(n+\frac{1}{2}\right)\theta\right).$$

For a general domain $\Omega \ni 0$, the eigenfunctions of $H^0_{1/2}$ are complex valued, and their nodal set consists of isolated points. However, one can define the antilinear operator K by $Ku = e^{i\frac{\theta}{2}}\overline{u}$, and say that u is K-real if $Ku = u$. Then,

- K commutes with $H^0_{1/2}$, so that one can choose a basis of K-real eigenfunctions of $H^0_{1/2}$;
- the nodal set of a K-real eigenfunction of $H^0_{1/2}$ has the same regularity properties as the nodal set of an eigenfunction of the Laplacian, except at 0, where an odd number of nodal lines meet.

The basis given above consists of K-real eigenfunctions.

A twofold Riemannian covering

We can adopt an alternative geometric point of view.
We define a twofold covering of the punctured disk $\dot{\mathbb{D}}$.

$$\begin{array}{rccc} \Pi : & \mathbb{D}_c \equiv (0,1) \times [0,4\pi) & \to & \dot{\mathbb{D}} \equiv (0,1) \times [0,2\pi) \\ & (r,\theta) & \mapsto & (r, \theta \bmod 2\pi). \end{array}$$

We define the deck map (an isometric involution of the covering)

$$\begin{array}{rccc} \sigma : & \mathbb{D}_c & \to & \mathbb{D}_c \\ & (r,\theta) & \mapsto & (r, \theta + 2\pi \bmod 4\pi). \end{array}$$

If we set $\Sigma u := u \circ \sigma$, the unitary operator Σ commutes with the Dirichlet Laplacian on $\mathbb{D}_c^2$. This implies that $-\Delta^D_{\mathbb{D}_c}$ preserves the spaces $\mathcal{S}$ and $\mathcal{A}$, of functions respectively symmetric and antisymmetric with respect to the deck map.
Since $L^2(\mathbb{D}_c^2) = \mathcal{S} \oplus \mathcal{A}$, the spectrum of $-\Delta^D_{\mathbb{D}_c^2}$ is the union (with multiplicities) of $(\lambda_j^S)_{j\geq 1}$, the spectrum of its restriction to $\mathcal{S}$, and $(\lambda_j^A)_{j\geq 1}$, the spectrum of its restriction to and $\mathcal{A}$. We have $\lambda_j^S = \lambda_j(\mathbb{D})$ and $\lambda_j^A = \lambda_j(0,1/2)$.
A K-real functions on $\dot{\mathbb{D}}$ is the projection of a function of the form $e^{i\frac{\theta}{2}} u$, where u is a real-valued antisymmetric function on $\mathbb{D}_c$.

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Aharonov-Bohm potential

For $\mathbf{X} = (X_1, \ldots, X_N) \in \mathbb{R}^{2N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$, we define

$$\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}(x) = \sum_{i=1}^{N} \alpha_i \mathbf{A}_1^0(x - X_i).$$

The following magnetic field is derived from $\mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}$:

$$\mathbf{B} = \mathrm{Curl}\, \mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}} = \partial_{x_1} A_2 - \partial_{x_2} A_1 = \sum_{i=1}^{N} \alpha_i \delta_{X_i}.$$

If B_i is a small disk centered at X_i, such that $X_j \notin B_i$ for $j \neq i$,

$$\alpha_i = \frac{1}{2\pi} \int_{B_i} \mathbf{B},$$

and therefore, for any closed path γ contained in $\mathbb{R}^2 \setminus \cup_{i=1}^{N} \{X_i\}$,

$$\oint_{\gamma} \mathbf{A}(s)\, ds = 2\pi \sum_{i=1}^{N} \mathrm{ind}_{\gamma}(X_i) \alpha_i.$$

Definition of Aharonov-Bohm operators

We define the magnetic Sobolev space

$$\mathcal{H}_0^1(\mathbf{X},\boldsymbol{\alpha}) := \left\{ u \in L^2(\Omega,\mathbb{C})\,;\,(i\nabla + \mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}})u \in L^2(\Omega,\mathbb{C}^2) \right\}.$$

It can be shown (using a Sobolev injection and a Hardy-type inequality) that $\mathcal{H}_0^1(\mathbf{X},\boldsymbol{\alpha})$ is compactly embedded in $L^2(\Omega,\mathbb{C})$.
We define the Aharonv-Bohm operator $H_{\boldsymbol{\alpha}}^{\mathbf{X}}$ as the Friedrichs' extension of the positive symmetric differential operator $\left(i\nabla + \mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}\right)^2$ acting on $C_c^\infty(\Omega_{\mathbf{X}},\mathbb{C})$, where $\Omega_{\mathbf{X}} := \Omega \setminus \{X_1,\dots,X_N\}$.
It is a positive self-adjoint operator. The previous remark tells us it has compact resolvent.
Its spectrum therefore consists of a sequence of positive eigenvalues having finite multiplicity and tending to $+\infty$. We denote it by $(\lambda_j(\mathbf{X},\boldsymbol{\alpha}))_{j\geq 1}$ (counted with multiplicities).
The eigenvalues can be defined alternatively using only the quadratic form

$$u \mapsto \int_\Omega \left| \left(i\nabla + \mathbf{A}_{\boldsymbol{\alpha}}^{\mathbf{X}}\right) u \right|^2$$

on $\mathcal{H}_0^1(\mathbf{X},\boldsymbol{\alpha})$ and the min-max formula.

Gauge transformations

Let $u \in L^2(\Omega, \mathbb{C})$, $\mathbf{A} \in C^\infty(\Omega, \mathbb{R}^2)$ and $\varphi \in C^\infty(\Omega, \mathbb{R})$. We set

$$u^* := e^{i\varphi} u \text{ and } \mathbf{A}^* = \mathbf{A} + \nabla\varphi.$$

Then $(i\nabla + \mathbf{A}^*)\, u^* = e^{i\varphi}\,(i\nabla + \mathbf{A})\, u$.
More generally, given a function $\psi \in C^\infty(\Omega_{\mathbf{X}}, \mathbb{C})$ such that $|\psi| = 1$, we define the associated gauge transformation of magnetic potentials by $\mathbf{A}^* = \mathbf{A} - i\frac{\nabla\psi}{\psi}$.
It is known that such a gauge transformation does not change the eigenvalues of a magnetic Schrödinger operator.

Proposition

Let $\mathbf{A} \in C^\infty(\Omega_{\mathbf{X}}, \mathbb{R}^2)$. It is gauge equivalent to 0 if, and only if,

$$\frac{1}{2\pi} \oint_\gamma \mathbf{A}(s)\, ds \in \mathbb{Z}$$

for any close path γ contained in $\Omega_{\mathbf{X}}$.

In the case of Aharonov-Bohm operators, two potentials whose flux differ by an integer give rise to the same eigenvalues. In particular, we can always remove the poles having integer flux.

Magnetic conjugation operator and nodal set

Let us assume that for $i \in \{1, \ldots, N\}$, $\alpha_i \in \frac{1}{2} + \mathbb{Z}$. Then the potential $2\mathbf{A}_{\alpha}^{\mathbf{X}}$ is gauge equivalent to 0. Let ψ be a gauge function sending $2\mathbf{A}_{\alpha}^{\mathbf{X}}$ to 0. We define the unitary antilinear operator K by

$$K\, u = \psi\, \overline{u}.$$

We say that $u \in L^2(\Omega, \mathbb{C})$ is K-real if $K\, u = u$.
A direct computation shows that

$$K \circ H_{\alpha}^{\mathbf{X}} = H_{\alpha}^{\mathbf{X}} \circ K.$$

We can therefore find an orthonormal basis of $L^2(\Omega, \mathbb{C})$ consisting of K-real eigenfunction of $H_{\alpha}^{\mathbf{X}}$.
Furthermore, if u is a K-real eigenfunction of $H_{\alpha}^{\mathbf{X}}$, its zero set is regular. It enjoys the same regularity as the nodal set of Laplacian eigenfunction outside of the poles, and an odd number of nodal lines meet at the poles.

Magnetic characterization of minimal partitions

Theorem (B. Helffer and T. Hoffmann-Ostenhof, 2013)

Let us assume that $\mathcal{D} = (D_1, \dots, D_k)$ is a minimal k-partition of $\Omega \subset \mathbb{R}^2$. There exist a finite number of points $X_1, \dots, X_N \in \mathbb{R}^2$ such that $\mathcal{D}$ is the nodal partition associated with an eigenfunction u of the operator $H_{\boldsymbol{\alpha}}^{\mathbf{X}}$, with

$$\mathbf{X} = (X_1 \dots, X_N)$$

and

$$\boldsymbol{\alpha} = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$$

Furthermore, the eigenfunction u is associated with the eigenvalue $\lambda_k(\mathbf{X}, \boldsymbol{\alpha})$,.

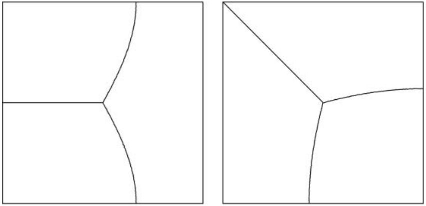
To build the magnetic potential, we have to add poles:

- at each singular point of the boundary of $\mathcal{D}$ where an odd number of curves meet,
- in each hole with an odd number of curves touching its boundary.

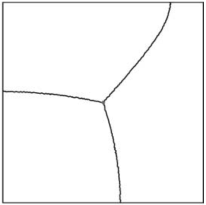
Example

From V. Bonnaillie-Noël, B. Helffer, and T. Hoffmann-Ostenhof. We have

$$\Lambda_3(\mathcal{D}) = \lambda_3^{AB}(C) \simeq 66.581.$$



(a) Candidates with different symmetries.



(b) Asymmetric candidate.

Moving poles

Theorem

For all integer $j \geq 1$ and all $\boldsymbol{\alpha} \in \mathbb{R}^N$, the function $\mathbf{X} \mapsto \lambda_j(\mathbf{X}, \boldsymbol{\alpha})$ is continuous in $\mathbb{R}^{2N}$.

Corollary

Let us assume that Ω that is simply connected, that $\alpha \in \mathbb{R}$, that $X_0 \in \partial\Omega$, and that $X \in \Omega$. Then, for any $j \geq 1$,

$$\lim_{X \to X_0} \lambda_j(X, \alpha) = \lambda_j(\Omega).$$

Corollary

Let us assume that $X_0 \in \Omega$ and that $X, Y \in \Omega$. Then, for any $j \geq 1$,

$$\lim_{X,Y \to X_0} \lambda_j((X, Y), (-1/2, 1/2)) = \lambda_j(\Omega).$$

In both cases, the limit operator is unitarily equivalent to the Dirichlet Laplacian, by a gauge transformation of the potential.

Plan

Setting and result

Let us assume that $0 \in \Omega$, let $\mathbf{a}_\pm(\varepsilon) := (\pm\varepsilon, 0)$ for $\varepsilon > 0$, and let us consider the Aharonov-Bohm operator $H_\varepsilon := H^{(\mathbf{a}_+(\varepsilon),\mathbf{a}_-(\varepsilon))}_{(1/2,-1/2)}$. We denote the eigenvalues by $(\lambda_j(\varepsilon))_{j\geq 1}$.
According to the continuity result, for any integer $j \geq 1$,

$$\lim_{\varepsilon\to 0^+} \lambda_j(\varepsilon) = \lambda_j(\Omega).$$

Theorem

Assume that the eigenvalue $\lambda_k(\Omega)$ is simple. Let u_k be an associated L^2-normalized eigenfunction. Assume additionally that $u_k(0) \neq 0$. Then

$$\lambda_k(\varepsilon) - \lambda_k(\Omega) \sim \frac{2\pi}{|\log(\varepsilon)|}|u_k(0)|^2 \text{ as } \varepsilon \to 0^+.$$

Outline of the proof

By continuity, $\lambda_k(\varepsilon)$ is a simple eigenvalue of H_ε for $\varepsilon > 0$ small enough. Let us denote by u_k^ε an associated eigenfunction.

Claim 1

There exists $r > 0$ and $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the zeroset of u_k^ε in $B(0, r)$ consists of a simple curve Γ_ε connecting $\mathbf{a}_-(\varepsilon)$ and $\mathbf{a}_+(\varepsilon)$.

Claim 2

For $a \in (0, a_0)$ small enough, $\lambda_k(\varepsilon) = \lambda_k(\Omega \setminus \Gamma_\varepsilon)$.

Proof.

Let $\mathbf{A}_\varepsilon := \mathbf{A}_{(1/2,-1/2)}^{(\mathbf{a}_+(\varepsilon),\mathbf{a}_-(\varepsilon))}$. We have

$$\frac{1}{2\pi} \oint_\gamma \mathbf{A}_\varepsilon(s)\, ds = 0$$

for any closed path γ contained in $\Omega \setminus \Gamma_\varepsilon$, so that $\mathbf{A}_\varepsilon$ is gauge equivalent to 0. This shows that $\lambda_k(\varepsilon)$ is an eigenvalue of the Dirichlet Laplacian in $\Omega \setminus \Gamma_\varepsilon$. A continuity argument shows that $\lambda_k(\varepsilon) = \lambda_k(\Omega \setminus \Gamma_\varepsilon)$ if $\varepsilon > 0$ is small enough. □

We are reduced to finding the asymptotic behavior of $\lambda_k(\Omega \setminus \Gamma_\varepsilon)$ as $a \to 0^+$.

Asymptotic expension for a Laplacian eigenvalue in a perforated domain

We use the following result, similar to the ones by G. Courtois (1995), and which generalizes formulas obtained by S. Ozawa (1981).
We still assume that Ω is an open, bounded and connected set with $0 \in \Omega$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact and connected sets. Let us assume that

$$\max_{y \in K_\varepsilon} |y| \to 0 \text{ as } \varepsilon \to 0,$$

which implies in particular that $\operatorname{diam}(K_\varepsilon) \to 0$ as $\varepsilon \to 0$. We write $\Omega_\varepsilon := \Omega \setminus K_\varepsilon$.

Theorem

Assume that the eigenvalue $\lambda_k(\Omega)$ is simple. Let u_k be an associated L^2-normalized eigenfunction. Assume additionally that $u_k(0) \neq 0$. Then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \frac{2\pi}{|\log(\varepsilon)|} |u_k(0)|^2 \text{ as } \varepsilon \to 0.$$

To finish the proof for the two-poles Aharonov-Bohm operator, it would be enough to show that $\operatorname{diam}(\Gamma_\varepsilon) \leq C\varepsilon$ as $\varepsilon \to 0^+$. We are actually only able to prove the weaker estimate $\log(\operatorname{diam}(\Gamma_\varepsilon)) \sim \log(\varepsilon)$, which is however sufficient.

Plan

Setting and result

We make the additional assumption that Ω is symmetric with respect to the axis $\{x_2 = 0\}$. We define the Aharonov-Bohm operator H_ε and its eigenvalues $(\lambda_j(\varepsilon))_{j\geq 1}$ as before.
Let us assume that $\lambda_k(\Omega)$ is simple. Let u_k be a real-valued L^2-normalized eigenfunction associated with $\lambda_k(\Omega)$, and let us assume that $u_k(0) = 0$.
Since u_k is a Laplacian eigenfunction, there exist an integer $m \geq 1$ (the order of 0 as a zero of u_k), $\alpha \in [0, \pi)$ and $C \neq 0$ such that

$$u_k(r\cos\theta, r\sin\theta) = C\, r^m \sin(\alpha - m\theta) + O\left(r^{m+1}\right).$$

Since Ω is symmetric and $\lambda_k(\Omega)$ simple, u_k is either symmetric or antisymmetric with respect to the transformation $(x_1, x_2) \mapsto (x_1, -x_2)$, and therefore

$$u_k(r\cos\theta, r\sin\theta) = C\, r^m \cos(m\theta) + O\left(r^{m+1}\right) \text{ if } \alpha = \frac{\pi}{2}$$

or

$$u_k(r\cos\theta, r\sin\theta) = -C\, r^m \sin(m\theta) + O\left(r^{m+1}\right) \text{ if } \alpha = 0.$$

Theorem

Let us assume that the above hypotheses are satisfied and that u_k is symmetric. Then

$$\lambda_k(\varepsilon) - \lambda_k(\Omega) \sim \pi\, C^2\, C_m\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+,$$

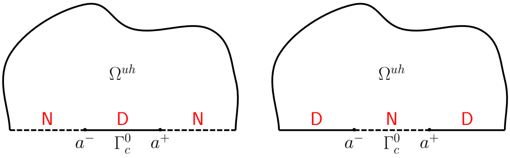
where C_m is a positive constant depending only on m and known explicitly.

An isospectrality result

Using the symmetry of Ω, we can define the antilinear unitary operator Σ by $\Sigma u = \overline{u} \circ \sigma$, where $\sigma : (x_1, x_2) \to (x_1, -x_2)$.
A direct computation shows that Σ commutes with H_ε. Therefore, H_ε leaves stable $\ker(\Sigma - I)$ (space of symmetric functions) and $\ker(\Sigma + I)$ (space of antisymmetric functions). The spectrum of H_ε can therefore by divided into symmetric and antisymmetric eigenvalues. We have, with multiplicities,

$$(\lambda_k(\varepsilon))_{k\geq 1} = \left(\lambda_k^S(\varepsilon)\right)_{k\geq 1} \cup \left(\lambda_k^A(\varepsilon)\right)_{k\geq 1}.$$

Using a suitable gauge transformation, we find that $\left(\lambda_k^S(\varepsilon)\right)_{k\geq 1}$ and $\left(\lambda_k^A(\varepsilon)\right)_{k\geq 1}$ are the symmetric eigenvalues of the Laplacian in Ω with additional Dirichlet conditions.



Using a continuity argument, we can show that, under the hypotheses of the theorem, $\lambda_k(\varepsilon)$ is a symmetric eigenvalue for $\varepsilon > 0$ small enough. We are reduced to studying the variation of $\lambda_k(\Omega)$ when we add a Dirichlet condition on a small segment.

Singular perturbation on the nodal set

Let us temporarily drop the symmetry hypothesis. We define the segment $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$.
We still assume that $\lambda_k(\Omega)$ is simple, and we still consider u_k a real-valued L^2-normalized associated eigenfunction. There exist $m \geq 1$, $\alpha \in [0, \pi)$ and $C \neq 0$ such that

$$u_k(r\cos\theta, r\sin\theta) = C\, r^m \sin(\alpha - m\theta) + O\left(r^{m+1}\right).$$

We assume that $\alpha \neq 0$, that is to say, that the segment s_ε is not tangent to a nodal line at 0. We $\Omega_\varepsilon := \Omega \setminus s_\varepsilon$.

Theorem

Let us assume the above hypotheses are satisfied. Then

$$\lambda_k(\Omega_\varepsilon) - \lambda_k(\Omega) \sim \pi\, C^2 \sin(\alpha)^2\, C_m\, \varepsilon^{2m} \text{ as } \varepsilon \to 0^+,$$

where C_m is the same as before.

To finish the proof for Aharonov-Bohm operators, we note that we were reduced to consider this type of perturbation in the case $\alpha = \frac{\pi}{2}$.