Review of spectral minimal partitions

Corentin Léna

Università degli Studi di Torino

ZiF Bielefeld, Mars 2017

Plan

Plan

A characterization of λ_2

Let Ω be a a bounded and connected open set in $\mathbb{R}^2$, or in a two-dimensional Riemannian manifold, with a sufficiently regular boundary.
We denote by $(\lambda_k(\Omega))_{k\geq 1}$ the eigenvalues, counted with multiplicity, of $-\Delta_\Omega^D$, the Dirichlet realization of the Laplacian in Ω.
For $D \subset \Omega$, open, we denote by $\lambda(D)$ the first eigenvalue of the Dirichlet Laplacian in D, i.e.

$$\lambda(D) = \inf_{u \in H_0^1(D)} \frac{\int_D |\nabla u|^2}{\int_D u^2},$$

and by u_D an associated normalized ground state.

Proposition

$\lambda_2(\Omega) = \mathfrak{L}_2(\Omega) := \inf \{\max(\lambda(D_1), \lambda(D_2)); D_1, D_2 \subset \Omega \text{ and } D_1 \cap D_2 = \emptyset\}.$

Proof of the characterization

Proof.

Step 1: $\lambda_2 \leq \mathfrak{L}_2(\Omega)$. For (D_1, D_2) admissible pair and $\alpha = (\alpha_1, \alpha_2)$, we set

$$u_\alpha = \alpha_1 \chi_{D_1} u_{D_1} + \alpha_2 \chi_{D_2} u_{D_2}.$$

We have

$$\int_\Omega u_\alpha^2 = \alpha_1^2 + \alpha_2^2 \text{ and } \int_\Omega |\nabla u_\alpha|^2 = \alpha_1^2 \lambda(D_1) + \alpha_2^2 \lambda(D_2).$$

We choose $\alpha \neq 0$ so that $\langle u_\alpha, u_\Omega \rangle = 0$. From the maxmin principle,

$$\lambda_2(\Omega) \leq \frac{\int_\Omega |\nabla u_\alpha|^2}{\int_\Omega u_\alpha^2} \leq \max(\lambda(D_1), \lambda(D_2)).$$

Step 2: $\mathfrak{L}_2(\Omega) \leq \lambda_2$. Let u_2 be an eigenfunction associated with λ_2. It has exactly two nodal domains, D_1^* and D_2^*, and $\lambda(D_i^*) = \lambda_2(\Omega)$ for $i \in \{1, 2\}$. □

Remark

Going back to step 1 in the proof above, we see that the minimal pairs of domains are exactly the pairs of nodal domains for a second eigenfunction.

Sum of eigenvalues

Definition

$\mathfrak{L}_{2,1}(\Omega) := \inf\left\{\frac{1}{2}\left(\lambda(D_1)+\lambda(D_2)\right); D_1, D_2 \subset \Omega \text{ and } D_1 \cap D_2 = \emptyset\right\}$.

- $\mathfrak{L}_{2,1}(\Omega) \leq \mathfrak{L}_2(\Omega)$. Indeed, for any admissible pair (D_1, D_2),

$$\frac{1}{2}\left(\lambda(D_1)+\lambda(D_2)\right) \leq \max(\lambda(D_1), \lambda(D_2)).$$

- If (D_1^*, D_2^*) is minimal for the sum and $\lambda(D_1^*) = \lambda(D_2^*)$, then it is also minimal for the maximum.

Proposition (Helffer–Hoffmann-Osthenhof, 2010)

If Tr is an equilateral triangle, $\mathfrak{L}_{2,1}(\mathrm{Tr}) < \mathfrak{L}_2(\mathrm{Tr})$.

Proposition (Helffer–Hoffmann-Osthenhof, 2010)

If there exists an eigenfunction u_2 in Ω, associated with the eigenvalue $\lambda_2(\Omega)$, with nodal domains D_1 and D_2 such that $\|u_2\|_{L^2(D_1)} \neq \|u_2\|_{L^2(D_2)}$, then $\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_2(\Omega)$.

First variation

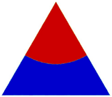
Let $\Gamma_{1,2}$ be the boundary between D_1 and D_2.
Let X be a smooth vector field supported in a neighborhood of the inside of $\Gamma_{1,2}$.
We define $\Phi_t(x) := x + tX(x)$, $D_i^t := \Phi_t(D_i)$ for $i \in \{1,2\}$, and

$$F(t) := \frac{1}{2}\left(\lambda(D_1^t) + \lambda(D_2^t)\right).$$

According to Hadamard formula,

$$F'(0) = \left(\frac{1}{\|u_2\|^2_{L^2(D_2)}} - \frac{1}{\|u_2\|^2_{L^2(D_1)}}\right)\int_{\Gamma_{1,2}} (\partial_\nu u_2)^2 \, X \cdot \nu.$$

Let us note that the result does not require that all the eigenfunctions associated with $\lambda_2(\Omega)$ are of this type.
We can find an explicit example for Tr.



Statement of the general problem

Definition

A k-partition of Ω is family $\mathcal{D} = (D_1, \ldots, D_k)$ of k open , connected and mutually disjoint subsets of Ω.

Definition

For $p \in [1, \infty]$, the p-energy of a k-partition $\mathcal{D} = (D_1, \ldots, D_k)$ is

$$\Lambda_{k,p}(\mathcal{D}) = \left(\frac{1}{k} \sum_{i=1}^{k} \lambda(D_i)^p \right)^{\frac{1}{p}} \text{ if } p < \infty \text{ and } \Lambda_{k,\infty}(\mathcal{D}) := \max_{1 \le i \le k} \lambda(D_i).$$

Definition

For $p \in [1, \infty]$, we define $\mathfrak{L}_{k,p}(\Omega) = \inf \{\Lambda_{k,p}(\mathcal{D}); \mathcal{D} \text{ } k\text{-partition of } \Omega\}$.
A k-partition $\mathcal{D}$ is called p-minimal if $\Lambda_{k,p}(\mathcal{D}) = \mathfrak{L}_{k,p}(\Omega)$.

Properties

- For $1 \le p < q \le \infty$, $\mathfrak{L}_{k,p}(\Omega) \le \mathfrak{L}_{k,q}(\Omega)$.
- Let us call a k-partition $\mathcal{D}$ equispectral if $\lambda(D_1) = \cdots = \lambda(D_k)$. If an equispectral partition is p-minimal, it is also q-minimal for $q \in (p, \infty]$.

Proof.

Use Hölder's inequality.

Existence and regularity

Existence and regularity follow from the work of several authors :Bucur, Buttazzo, and Henrot; Caffarelli and Lin; Conti, Terracini, and Verzini; Helffer, Hoffmann-Ostenhof, and Terracini. We refer to the results by Helffer, Hoffmann-Ostenhof, and Terracini (2009).

Assumption A

Ω has a piecewise-$C^{1,+}$ boundary and satisfies the interior cone property.

Theorem (existence)

Under assumption A, there exists a p-minimal k-partition of Ω.

Theorem (regularity)

Under assumption A, any p-minimal k-partition of Ω is regular (up to 0-capacity sets).

Proposition

An ∞-minimal k-partition is equispectral.

Remark

When $p = \infty$, we write Λ_k, $\mathfrak{L}_k$ and talk of minimal k-partitions.

More on regularity

We say that the k-partition $\mathcal{D} = (D_1, \ldots, D_k)$ is strong if

$$\Omega = \text{Int}\left(\cup_{i=1} \overline{D}_i\right) \setminus \partial\Omega.$$

In that case we define the boundary of $\mathcal{D}$ as $N(\mathcal{D}) := \overline{\cup_{i=1}^{k} \partial D_i \setminus \partial\Omega}$.
We say that $\mathcal{D}$ is regular if it is strong and its boundary $N := N(\mathcal{D})$ satisfies the following properties.

i. $\Omega \cap N$ is locally a $\mathcal{C}^{1,1-}$, curve except for the points in a finite set S_{int};

ii. to each $x \in S_{int}$ corresponds an integer $\nu(x) \geq 2$ such that N, in a neighborhood of x, consists of $v(x)$ half-curves of class $\mathcal{C}^{1,+}$ ending at x;

iii. $S_{bd} = \partial\Omega \cap N$ is finite and to each $z \in S_{bd}$ corresponds an integer $\rho(z) \geq 1$ such that, in a neighborhood of z, N consists of $\rho(z)$ half-curves of class $\mathcal{C}^{1,+}$ contained in $\overline{\Omega}$ and meeting $\partial\Omega$ at z;

iv. at each point in S_{int}, the half-curves make equal angles;

v. at each point in $N \cap \partial\Omega$, the half-curves and $\partial\Omega$ make equal angles.

Points iv. and v. will be called the equal angle meeting property.

An Euler formula for regular partitions

Assume that Ω is a open, bounded and connected set in $\mathbb{R}^2$ satisfying the assumption A.
Assume that $\mathcal{D}$ is a regular k-partition of Ω.
Let

- b_0 be the number of connected components of $\partial\Omega$;
- b_1 be the number of connected components of $\partial\Omega \cup N(\mathcal{D})$.

Then

$$k = b_1 - b_0 + 1 + \sum_{x \in S_{int}} \left(\frac{\nu(x)}{2} - 1 \right) + \sum_{z \in S_{bnd}} \frac{\rho(x)}{2}.$$

Let n_{odd} be the number of points x in S_{int} such that $\nu(x)$ is odd.
Then

$$\sum_{x \in S_{int}} \left(\frac{\nu(x)}{2} - 1 \right) \geq \frac{n_{odd}}{2}.$$

On the other hand

$$b_1 + \sum_{x \in S_{int}} \left(\frac{\nu(x)}{2} - 1 \right) \geq 1 + \frac{b_0}{2}.$$

Finally, $n_{odd} \leq 2k + b_0 - 4$. For a minimal partition $n_{odd} \leq 2k - 4$.

Nodal minimal partitions

Definition

We say that a k-partition $\mathcal{D} = (D_1, \ldots, D_k)$ is nodal if the D_i's are the nodal domain of an eigenfunction u of $-\Delta_\Omega^D$.

Definition

Given a regular k-partition $\mathcal{D} = (D_i)_{1 \leq i \leq k}$, we say that two domains D_i and D_j are neighbors if the set

$$D_{i,j} := \mathrm{Int}\left(\overline{D}_i \cup \overline{D}_j\right)$$

is connected.

Definition

We say that a regular partition is bipartite its domains with only two colors such that two neighbors have a different color.

A nodal partition is bipartite (associate a color to each sign of the eigenfunction). The converse is true for minimal partitions.

Theorem (Helffer–Hoffmann-Ostenhof–Terracini, 2009)

If a minimal k-partition of Ω is bipartite, then it is nodal.

Courant-sharp eigenfunctions

Theorem (Courant, 1923)

If u is an eigenfunction associated with $\lambda_k(\Omega)$, it has at most k nodal domains.

Definition

We say that u_k, eigenfunction associated with $\lambda_k(\Omega)$, is Courant-sharp when it has k nodal domains.

Proposition

Let $\mathcal{D} = (D_1, \ldots, D_k)$ be a spectral equipartition. Then $\lambda_k(\Omega) \leq \Lambda_k(\mathcal{D})$, with equality if, and only if, $\mathcal{D}$ is the nodal partition for an eigenfunction associated with $\lambda_k(\Omega)$.

Corollaries

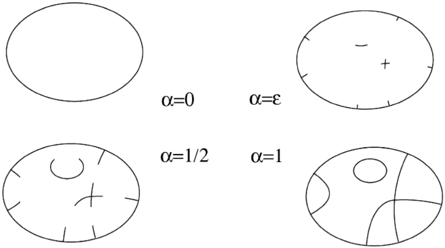
- $\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega)$.
- The nodal domains of a Courant-sharp eigenfunction give a minimal k-partition of Ω.
- If $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega)$, all minimal k-partitions are given by the nodal domains of a Courant-sharp eigenfunction.

Nodal minimal partitions

Theorem (Helffer–Hoffmann-Ostenhof–Terracini, 2009)

If a minimal k-partition of Ω is nodal, it is necessarily associated with $\lambda_k(\Omega)$.

To simplify notation, we set $\lambda_\ell := \lambda_\ell(\Omega)$ is this demonstration.
Let $=(D_1, \dots, D_k)$ be a minimal k-partition consisting of the nodal domains of u_m, eigenfunction associated with λ_m. We choose m such that $\lambda_m > \lambda_{m-1}$.
We define a family of open sets Ω_α, $\alpha \in [0, 1]$.



In particular, $\Omega_0 := \Omega$ and $\Omega_1 := \bigcup_{i=1}^{k} D_i$.
For $\alpha \in [0, 1]$, we denote by $(\lambda_\ell(\alpha))_{\ell \geq 1}$ the eigenvalues of $-\Delta^D_{\Omega_\alpha}$.

Claim 1

For every integer $\ell \geq 1$, $\alpha \mapsto \lambda_\ell(\alpha)$ is continuous and non-decreasing.

Claim 2

For every $\alpha \in [0, \alpha]$, λ_m is an eigenfunction of $-\Delta^D_{\Omega_\alpha}$ and u_m and associated eigenfunction.

Let us now assume, by contradiction, that $m > 1$.
Let us begin with the case where λ_m is simple.

Claim 3

There exist a smallest $\alpha_1 \in (0, 1)$ such that

$$\lambda_1(\alpha_1) < \lambda_2(\alpha_2) \leq \cdots \leq \lambda_{m-1}(\alpha_1) = \lambda_m(\alpha_1) = \lambda_m.$$

Proof.

$$\begin{array}{ll} \alpha = 0 & \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{m-1} < \lambda_m \\ \alpha = 1 & \underbrace{\lambda_m = \lambda_m = \cdots = \lambda_m = \lambda_m}_{k \text{ times}} \end{array}$$

□

We chose $v \in E(\alpha, \lambda_m)$ such that $\langle v, u_m \rangle_{L^2(\Omega)} = 0$.

Claim 4

There exists $\beta > 0$ (small) such that $w_\beta = u_m + \beta v$ has k nodal domains.

The nodal domains of w_β form a minimal bipartite k-partition of Ω, distinct from $\mathcal{D}$.
The function w_β is therefore an eigenfunction of $-\Delta^D_{\Omega_\alpha}$, contradicting the simplicity of λ_m.
Let us now consider the case $\lambda_m = \lambda_{m+1} = \lambda_{m+\ell-1} < \lambda_{m+\ell}$, with $\ell \geq 2$.
Claims 1-2-3 are still valid and proved in the same way.
We denote by $\underline{E}(\alpha_1, \lambda_m)$ the eigenfunction given by the restriction of elements in $E(0, \lambda_m)$ (for instance u_m).

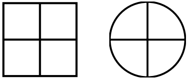
Claim 5

The inclusion $\underline{E}(\alpha_1, \lambda_m) \subset E(\alpha_1, \lambda_m)$ is strict.

We take $v \in E(\alpha_1, \lambda_m) \setminus \underline{E}(\alpha_1, \lambda_m)$ and proceed as before. We reach a contradiction by constructing w_β, and eigenfunction associated with λ_m which does not belong to $E(0, \lambda_m)$.

Applications

In the case of a square or a disk, $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4$, and there exists an eigenfunction u_4, associated with λ_4, with 4 nodal domains.
This tells us that minimal 3-partition are not nodal. The minimal 4-partitions are the following.



For the rectangle $\mathrm{R}(a, b) = (0, a) \times (0, b)$, the eigenvalues are

$$\lambda_{m,n}(a, b) = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \text{ for } m, n \geq 1.$$

Proposition

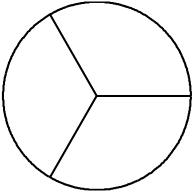
Let $a > b$ and $\frac{b^2}{a^2} \notin \mathbb{Q}$. Then the only cases when $\lambda_{m,n}(a, b)$ has a Courant-sharp eigenfunction are the following:

1. $(m, n) = (3, 2)$ if $\sqrt{\frac{8}{5}} < \frac{a}{b} < \sqrt{\frac{7}{3}}$;
2. $(m, n) = (2, 2)$ if $1 < \frac{a}{b} < \sqrt{\frac{5}{3}}$;
3. $(m, 1)$ if $\sqrt{\frac{m^2-1}{3}} < \frac{a}{b}$.

Plan

A 3-partition of the disk

The following partition $\mathcal{D}^*$ of the unit disk $\mathbb{D}$ is conjectured to be minimal.



The partition $\mathcal{D}^*$ is not bipartite and therefore not nodal.
Let us define $\Omega' := \mathbb{D} \setminus [0,1]$.
Then $\mathcal{D}^*$ is a nodal partition of Ω', associated with the eigenvalue $\lambda_3(\Omega') = j_{\frac{3}{2},1}^2$, and given by the Courant-sharp eigenfunction

$$u(r,\theta) := J_{\frac{3}{2}}\left(j_{\frac{3}{2},1} r\right) \sin\left(\frac{3\theta}{2}\right).$$

A magnetic Schrödinger operator

Let us define

$$u^*(r,\theta) = e^{i\frac{\theta}{2}} u(r,\theta).$$

Then u^* admits a smooth (complex valued) extension to the punctured disk $\dot{\mathbb{D}} := \mathbb{D} \setminus \{0\}$.
The function u^* satisfies the eigenvalue equation

$$(i\nabla + \mathbf{A}_0)^2 u^* = \lambda_3(\Omega') u^*,$$

with

$$\mathbf{A}_0(r,\theta) := \frac{1/2}{r}\mathbf{e}_\theta.$$

More generally, we can define the operator H_0 as the Friedrichs extension of the differential operator $(i\nabla + \mathbf{A}_0)^2$, acting on function in $C_c^\infty(\dot{\mathbb{D}})$. This is positive self-adjoint operator with compact resolvent.
Physically, H_0 is (one realization of) a Schrödinger operator with a singular magnetic flux π concentrated at 0, similar to the model studied by Y. Aharonov and D. Bohm (1959).

The spectrum of H_0 consists of positive eigenvalues, which we denote by $(\lambda_k^{AB}(0))_{k\geq 1}$.
In this particular example, the eigenvalues are $\left\{ j^2_{n+\frac{1}{2},\ell};\, n \in \mathbb{N} \text{ and } \ell \in \mathbb{N}^* \right\}$, each with multiplicity 2. An associated basis of eigenfunction is given by

$$u^c_{n,\ell}(r,\theta) = J_{n+\frac{1}{2}}\left(j_{n+\frac{1}{2},\ell}\, r\right) e^{i\frac{\theta}{2}} \cos\left(\left(n+\frac{1}{2}\right)\theta\right)$$

and

$$u^c_{n,\ell}(r,\theta) = J_{n+\frac{1}{2}}\left(j_{n+\frac{1}{2},\ell}\, r\right) e^{i\frac{\theta}{2}} \cos\left(\left(n+\frac{1}{2}\right)\theta\right).$$

In general, the eigenfunctions of H_0 are complex valued, and their nodal set consists of isolated points. However, one can define the antilinear operator K by $Ku = e^{i\frac{\theta}{2}}\overline{u}$, and say that u is K-real if $Ku = u$. Then,

- K commutes with H_0, so that one can choose a basis of K-real eigenfunctions of H_0;
- the nodal set of a K-real eigenfunction of H_0 has the same regularity properties as the nodal set of an eigenfunction of the Laplacian, except at 0, where an odd number of nodal lines meet.

The basis given above consists of K-real eigenfunctions.

A twofold Riemannian covering

We can adopt an alternative geometric point of view.
We define a twofold covering of the punctured disk $\dot{\mathbb{D}}$.

$$\begin{array}{rccc} \Pi: & \mathbb{D}_c \equiv (0,1)\times[0,4\pi) & \to & \dot{\mathbb{D}} \equiv (0,1)\times[0,2\pi) \\ & (r,\theta) & \mapsto & (r,\theta \bmod 2\pi). \end{array}$$

We define the deck map (an isometric involution of the covering)

$$\begin{array}{rccc} \sigma: & \mathbb{D}_c & \to & \mathbb{D}_c \\ & (r,\theta) & \mapsto & (r,\theta+2\pi \bmod 4\pi). \end{array}$$

If we set $\Sigma u := u\circ\sigma$, the unitary operator Σ commutes with the Dirichlet Laplacian on $\mathbb{D}_c^2$. This implies that $-\Delta^D_{\mathbb{D}_c}$ preserves the spaces $\mathcal{S}$ and $\mathcal{A}$, of functions respectively symmetric and antisymmetric with respect to the deck map.
Since $L^2(\mathbb{D}_c^2)=\mathcal{S}\oplus\mathcal{A}$, the spectrum of $-\Delta^D_{\mathbb{D}_c^2}$ is the union (with multiplicities) of $(\lambda_k^S)_{k\geq1}$, the spectrum of its restriction to $\mathcal{S}$, and $(\lambda_k^A)_{k\geq1}$, the spectrum of its restriction to and $\mathcal{A}$.
We have $\lambda_k^S=\lambda_k(\mathbb{D})$ and $\lambda_k^A=\lambda_k^{AB}(0)$.
A K-real functions on $\dot{\mathbb{D}}$ is the projection of a function of the form $e^{i\frac{\theta}{2}}u$, where u is a real-valued antisymmetric function on $\mathbb{D}_c$.

Aharonov-Bohm operators

For $\mathbf{X} = (X_1, \dots, X_N) \in \mathbb{R}^2$ (which we call poles), we define the multipolar magnetic potential

$$\mathbf{A}_{\mathbf{X}}(x) = \sum_{j=1}^{N} \mathbf{A}_0(x - X_j)$$

For a given domain Ω, we define the multipolar Aharonov-Bohm operator $H_{\mathbf{X}}$ with Dirichlet boundary condition, as the Friedrichs extension of the differential operator

$$(-i\nabla - \mathbf{A}_{\mathbf{X}}(x))^2,$$

acting on $C_c^\infty(\Omega_{\mathbf{X}})$, with $\Omega_{\mathbf{X}} := \Omega \setminus \{X_1, \dots, X_N\}$.
We denote by $(\lambda_k^{AB}(\mathbf{X}))_{k \geq 1}$ the associated eigenvalues.
We generalize the operator K in the following way. The vector field $2\mathbf{A}_{\mathbf{X}}$ has integer circulation along along any path contained in $\Omega_{\mathbf{A}}$. Therefore, there exists $\varphi : \Omega_{\mathbf{X}} \to \mathbb{R}$ (possibly multivalued) such that $e^{i\varphi}$ is univalued and $\nabla\varphi = 2\mathbf{A}_{\mathbf{X}}$. We define $Ku := e^{i\varphi}\overline{u}$.
As before, K commutes with $H_{\mathbf{X}}$, and we can find a base of K-real eigenfunctions.
We give a geometric interpretation, i.e. construct a Riemannian twofold covering $\Pi : \Omega_c \to \Omega_{\mathbf{X}}$ and identify $H_{\mathbf{X}}$ with the restriction of $-\Delta_{\Omega_c}^D$ to the space of antisymmetric function. The construction is however more abstract.

Magnetic characterization of minimal partitions

Theorem (Helffer and Hoffmann-Ostenhof, 2013)

Let us assume that $\mathcal{D} = \{D_1, \dots, D_k\}$ is a minimal k-partition of $\Omega \subset \mathbb{R}^2$, a bounded open set with piecewise C^1 boundary. There exist a finite number of points $X_1, \dots, X_N$ in $\mathbb{R}^2$ such that $\mathcal{D}$ is the nodal partition associated with a K-real eigenfunction u of the operator $H_{\mathbf{X}}$, with

$$\mathbf{X} = (X_1 \dots, X_N)$$

Furthermore, the eigenfunction u is associated with the eigenvalue $\lambda_k^{AB}(\mathbf{X})$.

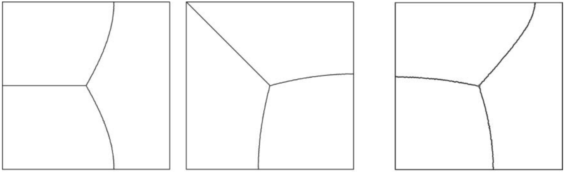
To build the magnetic potential, we have to add poles:

- at each singular point of the boundary of $\mathcal{D}$ where an odd number of curves meet;
- in each hole with an odd number of curves touching its boundary.

Example

From Bonnaillie-Noël, Helffer, and Hoffmann-Ostenhof (2009). When $\Omega = \mathrm{Sq}$, a square, it is conjectured that the minimal partition is obtained with one pole at the center C.
In that case, they show that $\lambda_3^{AB}(C)$ is multiple. We therefore have a one parameter family of three partition $\{\mathcal{D}^\theta; \theta \in \mathbb{S}^1\}$, such that

$$\Lambda_3(\mathcal{D}^\theta) = \lambda_3^{AB}(C) \simeq 66.581.$$



(a) Candidates with different symmetries. (b) Asymmetric candidate.

Let us note that they use the twofold covering approach in order to perform the numerical computation.
The conjecture $\mathfrak{L}_3(\mathrm{Sq}) = \lambda_3^{AB}(C)$ is supported by various numerical methods.

Application: minimal partitions for large k

Theorem (Helffer, 2015)

Let $(\mathcal{D}_k^*)_{k\geq 1}$ be a sequence of spectral minimal partitions of a good domain Ω. Let $n_{odd}^{(k)}$ be the number of interior singular points of $N(\mathcal{D}_k)$ of odd order. Then

$$\liminf_{k\to+\infty} \frac{n_{odd}^{(k)}}{k} \geq c_0 > 0.$$

The constant c_0 is explicit: if j is the first positive zero of the Bessel function J_0,

$$c_0 = \frac{\pi^3}{2^6 j^2}\left((j^4 + 10j^2 - 2) - 2(2j^2+1)\sqrt{1+2j^2}\right) \simeq 0.014.$$

The upper bound given by Euler formula is

$$\limsup_{k\to+\infty} \frac{n_{odd}^{(k)}}{k} \leq 2.$$

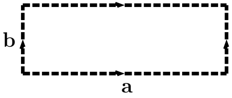
According to the Hexagonal Conjecture, one should have

$$\lim_{k\to+\infty} \frac{n_{odd}^{(k)}}{k} = 2.$$

Plan

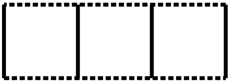
Statement of the problem for flat rectangular tori

For $0 < b \leq a$, we write $\mathrm{T}(a,b) := (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$.



Eigenvalues of $-\Delta_{\mathrm{T}(a,b)}$: $\lambda_{m,n}(a,b) = 4\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$, with $m, n \geq 0$.
There is a space of eigenfunctions $E_{m,n}(a,b)$ associated with $\lambda_{m,n}(a,b)$, with dimension 1, 2, or 4.
We define $\mathcal{D}_k(1,b)$ as the k-partition of $\mathrm{T}(1,b)$ with domains

$$D_i = \left(\frac{i-1}{k}, \frac{i}{k}\right) \times (0,b), \qquad \text{for } i = 1, \ldots, k.$$



Let us note that $\Lambda(\mathcal{D}_k(1,b)) = k^2\pi^2$. For which values of b is $\mathcal{D}_k(1,b)$ minimal.

k even

Proposition (Helffer-Hoffmann-Ostenhof, 2014)

If k is even, $\mathcal{D}_k(1,b)$ is minimal if, and only if, $b \leq 2/k$.

Proof.

We write $k = 2\ell$. Then $\mathcal{D}_k(1,b)$ is the nodal partition given by the eigenfunction

$$(x,y) \mapsto \sin(2\ell\pi x) \,,$$

associated with the eigenvalue $\lambda_{\ell,0}(1,b) = 4\ell^2\pi^2 = k^2\pi^2$.

According to the Courant-sharp characterization of nodal minimal partitions, $\mathcal{D}_k(1,b)$ is minimal if, and only if, $\lambda_{\ell,0}(1,b) = \lambda_k(T(1,b))$.

This occurs when

$$4\ell^2\pi^2 = \lambda_{\ell,0}(1,b) \leq \lambda_{1,b}(1,b) = \frac{4\pi^2}{b^2} \,,$$

i.e. when $b \leq 1/\ell = 2/k$. □

k odd

When the integer k is odd, the partition $\mathcal{D}_k(1,b)$ is not bipartite, and therefore not nodal.

Theorem (Helffer-Hoffmann-Ostenhof, 2014)

If k is odd and if $b < 1/k$, the partition $\mathcal{D}_k(1,b)$ is minimal.

Claim 1

If $D \subset T(1,b)$ is homeomorphic to a disk, $\lambda(D) \geq \frac{\pi^2}{b^2}$.

Proof.

We consider the covering

$$\begin{array}{cccc} \Pi_\infty : & \mathbb{R}^2 & \rightarrow & T(1,b) \\ & (x,y) & \mapsto & (x \bmod 1, y \bmod b). \end{array}$$

Let D_0 be one of the connected components of $\Pi_\infty^{-1}(D)$.
Let D_0^* be the Steiner symmetrization of D_0 with respect to $\{y=0\}$. It is contained in the strip $S_b = \mathbb{R}\times]-b/2, b/2[$.

By domain monotonicity of λ, $\frac{\pi^2}{b^2} \leq \lambda_1(D_0^*) \leq \lambda_1(D_0) = \lambda_1(D)$. □

We consider the following fourfold covering of $\mathrm{T}(1,b)$.

$$\begin{array}{rccc} \Pi: & T(2,2b) & \to & T(1,b) \\ & (x,y) & \mapsto & (x \bmod 1, y \bmod b). \end{array}$$

Claim 2

If $\mathcal{D}$ is a regular k-partition of $T(1,b)$ with no domain homeomorphic to a disk, $\Pi^{-1}(D)$ has 2 connected components for each domain D of $\mathcal{D}$. All these connected components form a $2k$-partition of $T(2,2b)$ denoted by $\Pi^{-1}(\mathcal{D})$, with the same energy as $\mathcal{D}$.

We can now finish the proof of the theorem.
Let $b < 1/k$ and let $\mathcal{D}^*$ be a minimal k- partition of $T(1,b)$.
$\mathcal{D}^*$ has no domain homeomorphic to a disk: otherwise, according to Claim 1, $\Lambda_k(\mathcal{D}^*) \geq \pi^2/b^2 > k^2\pi^2 = \Lambda_k(\mathcal{D}_k(1,b))$, contradicting the minimality.
According to Claim 2, $\mathcal{D}^*$ is lifted to a $2k$-partition $\Pi^{-1}(\mathcal{D}^*)$ of $\mathrm{T}(2,2b)$, with energy $\Lambda_k(\mathcal{D}^*)$. Since $2b/2 < 2/2k$, the analysis of the even case tells us that $\mathfrak{L}_{2k}(\mathrm{T}(2,2b)) = (2k)^2\pi^2/2^2 = k^2\pi^2$.
Therefore $\Lambda_k(\mathcal{D}^*) \geq k^2\pi^2 = \Lambda_k(\mathcal{D}_k(1,b))$, which conclude the proof.
Let us note additionally that $\Pi^{-1}(\mathcal{D}^*)$ is necessarily nodal, associated with $\lambda_{2k}(\mathrm{T}(2,2b))$. Up to a translation, it is $\mathcal{D}_{2k}(2,2b)$, and $\mathcal{D}^*$ is $\mathcal{D}_k(1,b)$.

We now look at the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

Theorem (Helffer–Hoffmann-Ostenhof–Terracini, 2014)

Any minimal 3-partition of $\mathbb{S}2$ is, up to a rotation, the **Y**-partition. In particular,

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}.$$

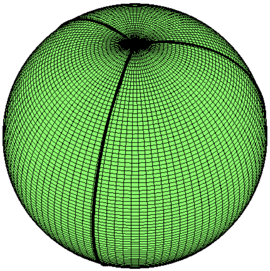


Figure : The **Y**-partition

We recall that the spectrum of $-\Delta_{\mathbb{S}^2}$ is the set $\{\ell(\ell+1); \ell \in \mathbb{N}\}$, and that $\ell(\ell+1)$ has multiplicity $2\ell+1$.

Claim 1

A minimal 3-partition of $\mathbb{S}^2$ is not nodal.

Proof.

We have $\lambda_2(\mathbb{S}) = \lambda_3(\mathbb{S}) = \lambda_4(\mathbb{S}) = 2$, and therefore $\lambda_3(\mathbb{S})$ has no Courant-sharp eigenfunction. □

Claim 2

A minimal 3-partition of $\mathbb{S}^2$ has the same topological type as the **Y**-partition: its boundary consists of two triple points connected by three non-crossing regular arcs.

Proof.

We use Euler's formula to enumerate all possible topological types, and exclude the bipartite ones. □

Claim 3

The boundary of a minimal 3-partition contains two antipodal points.

Proof.

This follows from a theorem of Lyusternick and Schnirelman, using Claim 2. □

Let us now fix $\mathcal{D}^* = (D_1, D_2, D_3)$ a minimal partition of $\mathbb{S}^2$, and two antipodal points x_+ and x_- in $N(\mathcal{D}^*)$.
We denote by $\Pi : \mathbb{S}^2_c \to \ddot{\mathbb{S}}^2$ the double covering of the punctured sphere $\ddot{\mathbb{S}}^2 = \mathbb{S}^2 \setminus \{x_+, x_-\}$ (it can be written explicitly using polar coordinates).
We define the deck map $\sigma : \mathbb{S}^2_c \to \mathbb{S}^2_c$ by $\sigma(z) \neq z$ and $\Pi(\sigma(z)) = \Pi(z)$. It is an isometric involution of $\mathbb{S}^2$.
For $i \in \{1, 2, 3\}$, $\Pi^{-1}(D_i)$ consists of two connected components. We denote by $\Pi^{-1}(\mathcal{D}^*)$ the corresponding 6-partition of $\mathbb{S}^2_c$.
The partition $\Pi^{-1}(\mathcal{D}^*)$ is pairwise symmetric, that is to say, for every domain D of $\Pi^{-1}(\mathcal{D}^*)$, $\sigma(D)$ is a domain of $\Pi^{-1}(\mathcal{D}^*)$, distinct form D.

Spectrum of the double covering

If we set $\Sigma u := u\sigma$, the unitary operator Σ commutes with the Laplacian on $\mathbb{S}^2_c$. This implies that $-\Delta_{\mathbb{S}^2_c}$ preserves the spaces $\mathcal{S}$ and $\mathcal{A}$, of functions respectively symmetric and antisymmetric with respect to the deck map.
Since

$$L^2(\mathbb{S}^2_c) = \mathcal{S} \oplus \mathcal{A},$$

the spectrum of $-\Delta_{\mathbb{S}^2_c}$ is the union (with multiplicities) of $(\lambda_k^S)_{k\geq 1}$, the spectrum of its restriction to $\mathcal{S}$, and $(\lambda_k^A)_{k\geq 1}$, the spectrum of its restriction to and $\mathcal{A}$.
We have $\lambda_k^S = \lambda_k(\mathbb{S}^2)$. On the other hand, $(\lambda_k^A)_{k\geq 1}$ consists of the set $\{\ell(\ell+1); \ell \in \mathbb{N} + 1/2\}$, with a multiplicity $2\ell + 1$ for $\ell(\ell+1)$.
We obtain

$$\lambda_1(\mathbb{S}^2_c) = 0,$$

$$\lambda_2(\mathbb{S}^2_c) = \lambda_3(\mathbb{S}^2_c) = \frac{3}{2},$$

$$\lambda_4(\mathbb{S}^2_c) = \lambda_5(\mathbb{S}^2_c) = \lambda_6(\mathbb{S}^2_c) = 2,$$

$$\lambda_7(\mathbb{S}^2_c) = \lambda_8(\mathbb{S}^2_c) = \lambda_9(\mathbb{S}^2_c) = \lambda_{10}(\mathbb{S}^2_c) = \frac{15}{4}.$$

In particular, $\lambda_6(\mathbb{S}^2_c)$ has no Courant-sharp eigenfunction, so the minimal 6-partitions of $\mathbb{S}^2_c$ are not nodal.

Pairwise symmetric partitions

We define

$$\mathfrak{L}_k^A(\mathbb{S}_c^2) = \inf\{\Lambda_{2k}(\mathcal{D}); \mathcal{D} \text{ pairwise symmetric } 2k\text{-partition of } \mathbb{S}_c^2\}.$$

Proposition

If $\mathcal{D}$ is a pairwise symmetric $2k$-partition, $\lambda_k^A \leq \Lambda_{2k}(\mathcal{D})$. Furthermore, equality occurs if, and only if, $\mathcal{D}$ is the nodal partition for an eigenfunction associated with λ_k^A.

We have seen $\lambda_3^A = 15/4$, and that $\Pi^{-1}(\mathcal{D}^*)$ is a pairwise symmetric partition. According to the previous list,

$$\mathfrak{L}_3(\mathbb{S}^2) = \Lambda_3(\mathcal{D}^*) = \Lambda_6(\Pi^{-1}(\mathcal{D}^*)) \geq \lambda_3^A = 15/4.$$

On the other hand, $\mathfrak{L}_3(\mathbb{S}^2) \leq 15/4$, since $15/5$ is the energy of the **Y**-partition. We therefore have

$$\mathfrak{L}_3(\mathbb{S}^2) = \frac{15}{4}.$$

Furthermore, $\Pi^{-1}(\mathcal{D}^*)$ is a nodal partition for an eigenfunction associated with λ_3^A, with 6 nodal domains. The partition $\mathcal{D}^*$ is therefore a rotation of the **Y**-partition.

Plan

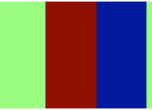
Numerical method for $T(1,b)$ (based on Bourdin–Bucur–Oudet, 2009)

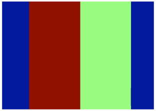
- Approximation of the max by an ℓ_p-norm:

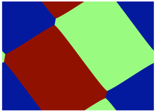
$$\Lambda_{k,p}(\mathcal{D}) = \left(\frac{1}{k}\sum_{i=1}^{k}\lambda_1(D_i)^p\right)^{\frac{1}{p}} \text{ with } p \in [1,\infty);$$

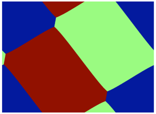
- Relaxation of the eigenvalue problem and penalization for implementing Dirichlet condition: for $\varepsilon > 0$ and $f : \mathrm{T}(1,b) \to [0,1]$, we define $\lambda(f,\varepsilon)$ as the first eigenfunction of $-\Delta + \frac{1}{\varepsilon}(1-f)$.
- Replacement of k-partitions by k-uples of functions $(f_1,\dots,f_k)$ satisfying $\sum_{i=1}^{k} f_i \equiv 1$.
- Discretization of the problem by a five points finite difference method for the Laplacian.
- Optimization by projected gradient.

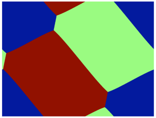
Numerical results for 3-partitions

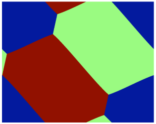

(a) $b = 0.70$

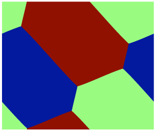

(b) $b = 0.71$

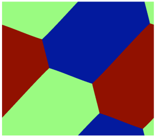

(c) $b = 0.72$

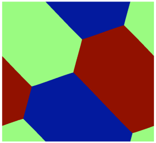

(d) $b = 0.73$

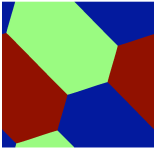

(e) $b = 0.76$

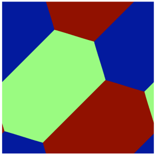

(f) $b = 0.80$


(g) $b = 0.84$


(h) $b = 0.88$


(i) $b = .92$


(j) $b = 0.96$


(k) $b = 1$

Theoretical results

Proposition

If k is an odd integer and if $0 < b \leq \frac{1}{\sqrt{k^2 - \frac{1}{8}}}$, $\mathcal{D}_k(1, b)$ is a minimal partition of $\mathrm{T}(1, b)$.

Proposition

If k is an odd integer and if $b > \frac{2}{\sqrt{k^2-1}}$, $\mathcal{D}_k(1, b)$ is a not minimal partition of $\mathrm{T}(1, b)$.

Conjecture: $2/\sqrt{k^2 - 1}$ is the "transition value".
This is supported by the numerical results.

Remark

If $0 < b \leq \frac{2}{\sqrt{k^2-1}}$ and $\mathcal{D}$ is a regular k-partition which is lifted to a $2k$-partition of $T(2, b)$, then

$$\Lambda_k(\mathcal{D}) \geq k^2\pi^2.$$